

ABUNDANT SOLUTIONS OF CERTAIN NONLINEAR EVOLUTION EQUATIONS ARISING IN SHALLOW WATER WAVES

PINKI KUMARI¹, R. K. GUPTA, AND SACHIN KUMAR

ABSTRACT. A family of new integrable Boussinesq equations with spatio temporal dimensions-(1+1) and (2+1) is studied in this work. The understudied equations are frequently used in computer models for the simulation of long water waves in shallow lakes and ocean harbours. Therefore, searching the exact travelling wave solutions of such equations are convenient in numerical as well as theoretical studies. In this work, a variety of travelling wave solutions i.e. Jacobi elliptic types, Weierstrass elliptic types, are obtained by the tanh function expansion method principle. Symbolic computations are made with the help of Maple software.

1. INTRODUCTION

The standard Boussinesq equation (BE) [1] is given by

$$(1.1) \quad u_{tt} - u_{xx} - a(u^2)_{xx} - bu_{xxxx} = 0,$$

introduced by Boussinesq to describe solution-interaction mechanism of shallow water waves in 1871. In fluid dynamics, the BE (1) combine several effects of waves and shallow water, including refraction, diffraction, shoaling and weak nonlinearity. Other than fluid dynamics, the equation (1.1) plays a crucial role in many fields of physics like vibrations in a nonlinear string, nonlinear lattice

¹corresponding author

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waves, and iron sound waves in a plasma, the shape-memory alloys, Rayleigh Benard convection. The BE (1.1) and its extensions have been extensively analyzed [3–7]. Wazwaz proposed new (2+1) and (3+1) dimensional integrable Boussinesq equations and obtained their multiple soliton solutions [8].

(1+1) dimensional Boussinesq equation

$$(1.2) \quad u_{tt} - u_{xx} - a(u^2)_{xx} - bu_{xxxx} + cu_{xt} = 0.$$

(2+1) dimensional Boussinesq equation

$$(1.3) \quad u_{tt} - u_{xx} - a(u^2)_{xx} - bu_{xxxx} + \frac{c^2}{4}u_{yy} + cu_{yt} = 0.$$

Here, the coefficients a, b, c, d are non zero constants. The above two models (1.2)-(1.3), describe the propagation of gravity waves over the water surface, more particularly, the head-on collision of oblique wave profiles, was proposed to examine the complete integrability [2].

The tanh function method is an direct, effective and powerful method for seeking exact solutions of a nonlinear system. The method is based upon the travelling wave hypothesis. So the method provides the solitary wave solutions in terms of finite order polynomial of tanh functions. The improvements of tanh method can be seen in [9–11].

In recent years, the symbolic computer program, maple, has made tedious and time consuming calculations easy and quick in context of differential equations. As a result, an exploration of directly searching for exact solutions of nonlinear dynamical systems has become an interesting and exciting research topic nowadays.

The study here will be focused on searching new exact solutions with distinct physical structures for the aforementioned three models (1.2)-(1.3) by using tanh function method principle.

The main steps of the algorithm to obtain the exact travelling wave solutions of the under-considered models is discussed in the following section.

2. ALGORITHM

Consider a nonlinear partial differential equation in the following form:

$$(2.1) \quad F(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots) = 0,$$

with independent variable (x, t) and dependent variable u .

Step1: Transform (1.1) by travelling wave transformation

$$u(x, t) = U(\xi), \quad \xi = k_1x + k_2t + p$$

into following ordinary differential equation:

$$(2.2) \quad G(U, U', U'', \dots) = 0,$$

where $'$ denotes the derivative w.r.t. ξ .

Step2: Assume the solution of (2.2) in the finite series form

$$(2.3) \quad U(\xi) = \sum_{i=0}^m a_i f^i(\xi),$$

where a_i are real constants, m is a positive integer and all to be determined later. Here, f can be chosen one among trigonometric functions, hyperbolic functions, Weierstrass P function, Jacobi elliptic functions.

Step3: Determine m by balancing principle.

Step4: Insert (2.3) into (2.2) and then by equating the coefficients of resulted polynomial in f to zero, we get a system of algebraic equations which, solving by maple, yield parameters a_i, k_1, k_2, p .

Step5: We can obtain the exact solutions of (2.1) from the result obtained in previous step.

3. RESULTS

In this section, a variety of the exact travelling wave solutions with distinct and rich physical structure for three models (1.2)-(1.3) of special interest in physics are demonstrated. It should be noted that abundant wave solutions, which have attracted much attention, are presented in a unified manner for the first time to the best of our knowledge.

3.1. Exact solutions of (1.2).

- Weierstrass P type solution

$$u(x, t) = \frac{ck_1k_2 - k_1^2 + k_2^2}{2ak_1^2} - \frac{6bk_1^2 \text{Weierstrass}P^2(k_1x + k_2t + p, c_1, c_2)}{a}$$

- Jacobi elliptic function type solutions

- CN type solution

$$u(x, t) = -\frac{8bc_1^2k_1^4 - 4bk_1^4 - ck_1k_2 + k_1^2 - k_2^2}{2ak_1^2} + \frac{6bc_1^2k_1^2 \text{Jacobi}CN^2(k_1x + k_2t + p, c_1)}{a}$$

- DN type solution

$$u(x, t) = \frac{4bc_1^2k_1^4 - 8bk_1^4 + ck_1k_2 - k_1^2 + k_2^2}{2ak_1^2} + \frac{6bk_1^2 \text{Jacobi}DN^2(k_1x + k_2t + p, c_1)}{a}$$

- NC type solution

$$u(x, t) = -\frac{8bc_1^2k_1^4 - 4bk_1^4 - ck_1k_2 + k_1^2 - k_2^2}{2ak_1^2} + \frac{6b(c_1^2 - 1)k_1^2 \text{Jacobi}NC^2(k_1x + k_2t + p, c_1)}{a}$$

- ND type solution

$$u(x, t) = \frac{4bc_1^2k_1^4 - 8bk_1^4 + ck_1k_2 - k_1^2 + k_2^2}{2ak_1^2} + \frac{6b(c_1^2 - 1)k_1^2 \text{Jacobi}ND^2(k_1x + k_2t + p, c_1)}{a}$$

- NS type solution

$$u(x, t) = \frac{4bc_1^2k_1^4 + 4bk_1^4 + ck_1k_2 - k_1^2 + k_2^2}{2ak_1^2} - \frac{6bk_1^2 \text{Jacobi}NS^2(k_1x + k_2t + p, c_1)}{a}$$

- SN type solution

$$u(x, t) = \frac{8bc_1^2k_1^4 - 4bk_1^4 - ck_1k_2 + k_1^2 - k_2^2}{2ak_1^2} - \frac{6b(c_1^2 - 1)k_1^2 \text{Jacobi}SN^2(k_1x + k_2t + p, c_1)}{a}$$

3.2. Exact solutions of (1.3).

- Weierstrass P type solution

$$u(x, y, t) = \frac{k_2^2c^2 + 4ck_2k_3 - 4k_1^2 + 4k_2^2}{8ak_1^2} - \frac{6bk_1^2 \text{Weierstrass}P^2(k_1x + k_2y + k_3t + p, c_1, c_2)}{a}$$

- Jacobi elliptic function type solutions

– CN type solution

$$u(x, y, t) = - \frac{32bc_1^2k_1^4 - 16k_1^4b - k_2^2c^2 - 4k_2k_3c + 4k_1^2 - 4k_3^2}{8ak_1^2} + \frac{6bc_1^2k_1^2 \text{JacobiCN}^2(k_1x + k_2y + k_3t + p, c_1)}{a}$$

– DN type solution

$$u(x, y, t) = \frac{16bc_1^2k_1^4 - 32k_1^4b + k_2^2c^2 + 4k_2k_3c - 4k_1^2 + 4k_3^2}{8ak_1^2} + \frac{6bc_1^2k_1^2 \text{JacobiDN}^2(k_1x + k_2y + k_3t + p, c_1)}{a}$$

– NC type solution

$$u(x, y, t) = - \frac{16bc_1^2k_1^4 - 32k_1^4b + k_2^2c^2 + 4k_2k_3c - 4k_1^2 + 4k_3^2}{8ak_1^2} + \frac{6b(c_1^2 - 1)k_1^2 \text{JacobiNC}^2(k_1x + k_2y + k_3t + p, c_1)}{a}$$

– ND type solution

$$u(x, y, t) = - \frac{16bc_1^2k_1^4 - 32k_1^4b + k_2^2c^2 + 4k_2k_3c - 4k_1^2 + 4k_3^2}{8ak_1^2} - \frac{6b(c_1^2 - 1)k_1^2 \text{JacobiND}^2(k_1x + k_2y + k_3t + p, c_1)}{a}$$

– NS type solution

$$u(x, y, t) = \frac{16bc_1^2k_1^4 + 16k_1^4b + k_2^2c^2 + 4k_2k_3c - 4k_1^2 + 4k_3^2}{8ak_1^2} - \frac{6k_1^2 \text{JacobiNS}^2(k_1x + k_2y + k_3t + p, c_1)}{a}$$

• SN type solution

$$u(x, y, t) = \frac{16bc_1^2k_1^4 + 16k_1^4b + k_2^2c^2 + 4k_2k_3c - 4k_1^2 + 4k_3^2}{8ak_1^2} - \frac{6b(c_1^2 - 1)k_1^2 \text{JacobiSN}^2(k_1x + k_2y + k_3t + p, c_1)}{a}$$

Note. Some new exact travelling wave solutions are presented, which are not reported before in the context of under-considered models, and can be helpful in getting better insights about simulation of water waves in shallow lakes and beaches..

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DEPARTMENT OF MATHEMATICS AND STATISTICS

CENTRAL UNIVERSITY OF PUNJAB, BATHINDA-15100, PUNJAB, INDIA

E-mail address: yadav.pk1403@gmail.com

DEPARTMENT OF MATHEMATICS

CENTRAL UNIVERSITY OF HARYANA, MAHENDERGARH-123031, HARYANA, INDIA

E-mail address: rajeshateli@gmail.com

DEPARTMENT OF MATHEMATICS AND STATISTICS

CENTRAL UNIVERSITY OF PUNJAB, BATHINDA-15100, PUNJAB, INDIA

E-mail address: sachin1jan@yahoo.com