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A NOTE ON NILPOTENT MODULES OVER RINGS

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ABSTRACT. In this paper, we define nilpotent elements and nilpotent submodules of a module M over a commutative ring R. We show that if R has the ascending chain condition, then the singular submodule $Z(M)$ is nil a nil submodule of M. Also we prove that if M is completely semi-prime module and the ring R has the ascending chain condition (a.c.c) on annihilators, then M contains no non-zero nil submodule.

1. INTRODUCTION

Throughout this paper all modules considered are left modules not necessarily unital. A non-zero element a of a ring R is nilpotent if $a^n = 0$ for some positive integer n. If R is commutative, then the set of all nilpotent elements forms an ideal called Nilradical of R which coincides with the intersection of all prime ideals of R. An ideal $I(\neq 0)$ of the ring R is nilpotent if $I^n = 0$ for some positive integer n. If every element of an ideal is nilpotent, then it is called nil ideal. It is evident that a nilpotent ideal is a nil ideal. However the converse is not true. Herstien and Small [6] shows that nil rings satisfying certain chain conditions are nilpotent. C Lanski [8] proves that the nil sub rings of Goldie rings [4] are nilpotent. Chatters and Hajarnavis [2, p.6] shows that a semi-ring with the a.c.c for right annihilators contains no non-zero nil one-sided ideals.

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For some basic definitions and results of rings and modules, we would like to mention Lambeck [7]. For a subset N of a module M, the set $Ann(N) = \{r \in R | rn = 0$ for any $n \in N\}$ is called the annihilator of N. If M is a module over a commutative ring R and N is a submodule, then $Ann(N)$ is an ideal of R. A submodule S of a module M is a prime submodule if for any sunmodule S_i of S, $Ann(S_i) = Ann(S)$. If M is a prime module over a commutative ring R with unity, then the ideal $Ann(M)$ of R is a prime ideal [1,3].

Recall that, a ring R is an Artinian(Noetherian) ring if it satisfies the ascending(descending) chain condition for its ideals. An ideal I of a ring R is an essential ideal if $I \cap J \neq (0)$ for any non-zero ideal J of R. The ideal J is a relative complement of I if it is maximal with respect to the property $I \cap J = (0)$. If the ideal J is the relative complement of the ideal I, then $I \bigoplus J$ is an essential ideal of the ring R [5, p.17]. A module M is a completely semi-prime if for any $m \in M$, $r^2 m = 0$ implies $rm = 0$ for any $r \in R$ [9]. M is a simple module if it has no proper submodules. The set $Z(M) = \{x \in M | Ix = 0$ for some essential ideal I of $R\}$ is a submodule of M known as singular submodule.

Lemma 1.1. *Let M be module over a commutative ring R with unity such that $Z(M) = 0$. Then M is completely semi-prime.*

Proof. Let I be any ideal of R such that $I^2 m = 0$, for some $m \in M$. Clearly $I \bigoplus J$ is an essential ideal for the relative complement J of I. Now $(I+J)I \subseteq I^2 + IJ$ gives that $(I+J)Im \subseteq I^2 m + IJm = 0$ as $I^2 m = 0$, $IJm = 0$. Thus $Im \subseteq Z(M)$ which gives that $Im = 0$. □

2. Nilpotent Elements and Nilpotent Submodules

Definition 2.1. *An element $m(\neq 0) \in M$ is a nilpotent element with index $k > 1$ if there exist a proper ideal I of R such that $I^k m = 0$ and $I^{k-1} m \neq 0$.*

A submodule $S(\neq 0)$ of the module M is nilpotent if there exists some proper ideal I such that $I^{k-1} S \neq 0$ but $I^k S = 0$ for some $k > 1$. If every element of a module M is nilpotent, then M is a Nil module.

Lemma 2.1. *If $m \in M$ is a nilpotent element, then $Ann(m)$ is not a prime ideal of R.*

Proof. Assume that $Ann(m)$ is a prime ideal. Let I be a proper ideal of R such that $Im \neq 0$ and $I^k m = 0$ for some $k > 1$. Then $I^k \subseteq Ann(m)$ gives that $I \subseteq Ann(m)$, a contradiction. □

Theorem 2.1. *If P is a prime submodule of the module M and $m \in P$ is a nilpotent element, then $Ann(P) \subset Ann(m)$.*

Proof. Since $m \in P$, we get $Ann(P) \subseteq Ann(m)$. Here $Ann(P) \neq Ann(m)$ for otherwise $Ann(m)$ will become a prime ideal [Lemma 2.5], a contradiction. □

Below we mention some examples.

Example 1. *The ring Z_4 of integer modulo 4 is a nilpotent module over Z since for the ideal $I =< 2 >$, $IZ_4 \neq 0$ but $I^2 Z_4 = 0$. Every elements of Z_4 is a nilpotent element. The ring Z_6 of integer modulo 6 is not a nilpotent module over Z_6. The proper ideals of Z_6 are $I_1 = \{0, 2, 4\}$, $I_2 = \{0, 3\}$ and $I_1^k Z_6 \neq 0$, $I_2^k Z_6 \neq 0$ for any $k \geq 0$. Also Z_6 contains no non-zero nilpotent element.*

Example 2. *The ring Z_p of integer modulo some prime p is a non-nilpotent module over the ring of integers Z. The module Z_p contains no nilpotent element.*

Example 3. *The Z-module $Z_{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}}$, where $p_1, p_2, \dots, p_n$ are distinct primes is a torsion module. The module $Z_{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}}$ contains no non-zero nilpotent elements for $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1$. However it contains nilpotent elements for at least one of $\alpha_i \geq 2$, $i = 1, 2, \dots, n$.*

3. Nilpotency in the context of Simple and Prime Modules

In this section, we deal with some sufficient conditions for existence of nilpotent elements and nilpotent submodules in a module of some special kind in particular prime, completely semi-prime and simple modules.

Lemma 3.1. *Let the ring R be commutative with unity and let $m \in M$ be a nilpotent element such that $Ann(m)$ is maximal as Annihilator. Then rm is also nilpotent for any $r \in R$.*

Proof. Let I be a proper ideal of R such that $Im \neq 0$ but $I^k m = 0$ for some positive integer $k > 1$. Then for any $r \in R$, $I^k(rm)$=0. Here $I(rm) \neq 0$, otherwise $I \subseteq Ann(rm) = Ann(m)$ as $Ann(m)$ is maximal as annihilator, a contradiction. □

Theorem 3.1. *If M is a simple module over the ring R, then M contains no non-zero nilpotent element*

Proof. Let $m(\neq 0) \in M$ be a nilpotent element. Let I be a proper ideal of R such that $Im \neq 0$ and $I^k m = 0$ for some $k > 1$. Now if $Im = M$, there exist some element $i(\neq 0) \in I$ such that $im = m$. We see that $i^k m = i^{k-1} im = i^{k-1} m$. Similarly $i^{k-1} m = i^{k-2}.im = i^{k-2} m$. Continuing in this way, we get $i^k m = m$, a contradiction. Thus Im is a proper submodule of M, a contradiction. □

In particular, if M is a simple module, then M is not nilpotent.

Theorem 3.2. *Let the ring R be Artinian and N be a submodule of M. If $m + I(\neq 0) \in \frac{M}{N}$ is nilpotent, then $m \in M$ is nilpotent.*

Example 4. *In the module $\frac{Z_2[x]}{<x^3>}$ over the non artinian ring $Z_2[x]$, the element $x+ < x^3 >$ is nilpotent since for the ideal $I =< x >$, $I(x+ < x^3 >) \neq 0$ but $I^2(x+ < x^3 >) = 0$. However x is not a nilpotent element of the module $Z_2[x]$ over itself.*

Theorem 3.3. *Let M be a non-nilpotent module over a Noetherian ring R and N be a submodule of M. Then $\frac{M}{N}$ is non-nilpotent.*

However the converse is of the above is not true.

Example 5. *The ring Z_4 of integer modulo 4 is a nilpotent module over the Noetherian ring Z, But the quotient $\frac{Z_4}{\{0,2\}}$ is a non-nilpotent module over Z. There exists no proper ideal I in Z such that $I\{0,2\} \neq 0$ and $I^k \frac{Z_4}{\{0,2\}} = 0$ for some $k > 1$.*

Theorem 3.4. *If M be a non-nilpotent module over a Noetherian ring R, then M contains no non-zero nilpotent element.*

Theorem 3.5. *Let M be a completely semi prime module over a commutative ring R with unity satisfying the a.c.c. on annihilators. Then M has no non-zero nil submodule.*

Proof. Let N be a non-zero submodule of the module M and let $n(\neq 0) \in N$ be such that $Ann(n)$ is maximal. Let $r(\neq 0) \in R$ be such that $r^2 n \neq 0$. For, if $r^2 n = 0$ for all $r \in R$ gives that $r \in Ann(rn) = Ann(n)$, a contradiction. Further $rn \neq 0$ for all $r \in R$. For otherwise $r^2 n = 0$, a contradiction. We claim that $n^{/} = rn$ is not a nilpotent element. Let I be a proper ideal of R such that

$In^/ \neq 0$. Then $I \nsubseteq Ann(n^/) = Ann(In^/) = Ann(n)$ giving thereby $I^2n^/ \neq 0$. Let $i \in I$ be such that $i^2n^/ \neq 0$ so that $Ann(n) = Ann(n^/) = Ann(i^2n^/)$. Now $in^/ \neq 0$ implies $i \notin Ann(i^2n^/)$. Thus $i \notin Ann(n^/) = Ann(i^2n^/)$ gives that $i^3n^/ \neq 0$. Hence $I^3n^/ \neq 0$ and so on. Thus for any $k > 1$, $I^kn^/ \neq 0$. □

4. Radical of Modules and its Nilpotency

Definition 4.1. *The intersection of all prime submodules of the module M is the prime radical of M denoted $Rad_P(M)$. If $M^/$ be the set of all nilpotent elements in a module M, then the submodule generated by $M^/$ is the Generalised Radical of M denoted $Rad_G(M)$.*

Theorem 4.1. *If M contains a nilpotent element m so that $Ann(m)$ is minimal as annihilator. Then $Rad_G(M)$ is also nilpotent.*

Theorem 4.2. *Let the ring R be commutative with the acc on annihilators. Then $Z(M)$ is a nil submodule of M.*

Proof. Let $m \in Z(M)$ and I be a proper ideal of R such that $Im \neq 0$. Consider the chain $Ann(Im) \subseteq Ann(I^2m) \subseteq Ann(I^3m) \subseteq ....$ Let $k > 1$ be such that $Ann(I^km) = Ann(I^{k+1}m)$. Suppose that $I^{k+1}m \neq 0$ and let $a \in I$ with $I^kam \neq 0$ such that $Ann(am)$ is maximal. Let $i \in I^k$ be such that $iam \neq 0$. Let $b \in I$ be any element so that $bm \in Z(M)$. Now $Ann(bm) \cap < ia > \neq 0$ since $Ann(bm)$ is an essential ideal. Let $x(\neq 0) \in Ann(bm) \cap < ia >$. Then $xbm = 0$ and $x = ria(\neq 0)$ for some $r \in R$. Thus $x \in Ann(abm)$ but $x \notin Ann(am)$. For if $xam = 0$, we get $ria.am = 0$ implies that $i \in Ann(ra^2m) = Ann(am)$, a contradiction. Thus $Ann(am) \nsubseteq Ann(abm)$. Hence by the choice of a, we get that $I^kbam = I^kabm = 0$, a contradiction to $I^{k+1}m \neq 0$. Hence m is nilpotent. □

Theorem 4.3. *Let the ring R be commutative with unity satisfying the acc on annihilators. Then the following statements are equivalent.*

(i) *M is completely semi-prime*
(ii) *M has no non-zero nil submodules*
(iii) *$Z(M) = 0$.*

Proof. If M is completely semi-prime, then M has no non-zero nil submodules. Assume that M contains no non-zero nil submodules. Let $r \in R$ be such that $r^2m = 0$ but $rm \neq 0$. Consider the ideal $I =< r >$ and submodule $< m >$.

Clearly $I < m > \neq 0$ and $I^2 < m >= 0$. Thus $< m >$ is a non-zero nil submodule, a contradiction. Next assume that M has no non-zero nil submodules. Thus $Z(M) = 0$ since $Z(M)$ is a nil submodule. Finally assume that $Z(M) = 0$. Let $m(\neq 0) \in M$ be such that for some $r \in R$, $r^2 m = 0$ but $rm \neq 0$. Consider the ideal $I =< r >$ and J be a relative complement of I. Then $I \bigoplus J$ is an essential ideal of R. Also $(I + J)Im = 0$ since $I^2 = 0$ and $IJ = 0$. Thus $Im \subseteq Z(M)$. But $Im \neq 0$, a contradiction. $\square$

Example 6. *The direct product $M = Z[x]X\frac{Z[x]}{<p>}$, for a prime p is an abelian group with respect to the operation addition defined as*

$$(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 +_p n_2),$$

where $m_1, m_2 \in Z[x], n_1, n_2 \in \frac{Z[x]}{<p>}$. Clearly M is a module over the integral domain Z. $M_1 = \{(m, 0) \in M | m \in Z[x], 0 \in \frac{Z[x]}{<p>}\}$ and $M_2 = \{(0, n) \in M | 0 \in Z[x], n \in \frac{Z[x]}{<p>}\}$ are submodules of M. The annihilators of M_1 and M_2 are 0 and $< p >$ respectively.

Theorem 4.4. *Let R be an integral domain and $m \in M$ be such that $Ann(m) \neq 0$. Then m is not a nilpotent element implies that $m \notin Rad_P(M)$.*

Proof. Let I be any proper ideal of R such that $Im \neq 0$. Then for any $k > 1$, $I^k m \neq 0$. Consider $T = \{I^k Ann(m) | k \geq 0\}$. Clearly $(0) \notin T$. Let P be a submodule such that $Ann(P)$ is a maximal ideal with $I^k Ann(m) \cap Ann(P) = \{0\}$ for any $k \geq 0$. We claim that $Ann(P)$ is a prime ideal. Let $a, b \notin Ann(P)$ be two elements. Then $(Ann(P) + aR) \cap I^k Ann(m) \neq 0$ and $(Ann(P) + bR) \cap I^s Ann(m) \neq 0$ for some $k, s \geq 0$. Thus we get that $(Ann(P) + aR)(Ann(P) + bR) \cap I^k Ann(m) I^s Ann(m) \neq 0$. For, let $x(\neq 0) \in (Ann(P) + aR) \cap I^k Ann(m)$ and $y(\neq 0) \in (Ann(P) + bR) \cap I^s Ann(m)$ gives $xy(\neq 0) \in (Ann(P) + aR)(Ann(P) + bR)$ and $xy \in I^k Ann(m) I^s Ann(m)$. Thus $(Ann(P) + abR) \cap I^k Ann(m) \neq 0$ as $((Ann(P) + aR)(Ann(P) + bR) \cap I^k Ann(m) I^s Ann(m)) \subseteq (Ann(P) + abR) \cap I^k Ann(m)$. Now if $ab \in Ann(P)$, then $Ann(P) \cap I^k Ann(m) \neq 0$, a contradiction. Thus $Ann(P)$ is a prime ideal. Now for any submodule $S(\neq 0)$ of P, then $Ann(P) = Ann(S)$. Hence P is a prime submodule. Again let $m \in P$. Then $Ann(P) \subseteq Ann(m)$. Thus $Ann(m) \cap Ann(P) = Ann(P)(\neq 0)$, a contradiction. Thus $m \notin P$. Hence $m \notin Rad_P(M)$. $\square$

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