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ON DUAL MULTIPLIERS IN CI-ALGEBRAS

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ABSTRACT. The Concept of BCK/BCI-algebras was first introduced by Y. Imai and K. Iseki [2,3] in 1966. These BCK/BCI algebras can be generalized into several different categories of algebras like BCH [1], BH [4], d [9], etc. Later on, dual BCK algebras [5] was introduced which paved the way for development of BE-algebras [6]. In 2010, B. L. Meng [8] introduced the idea of CI-algebras as a generalization of BE-algebras which is considered to be an important algebraic structure till date. The concept of Cartesian product has been developed in 2013 [10] which plays a key role in the development of this CI-algebras. A new concept of Absorptive CI-algebra has been developed in 2016. The idea of Multipliers in BE-algebras [7] has been utilized to develop the idea of Multipliers in CI-algebras [12] in 2019. In this paper we present, definition of Dual Multiplers in CI-algebras and talk about few examples, characteristics of this map.

1. INTRODUCTION

The Concept of BCK/BCI-algebras was first introduced by Y. Imai and K. Iseki [2, 3] in 1966. These BCK/BCI algebras can be generalized into several different categories of algebras like BCH [1], BH [4], d [9], etc. Later on, dual BCK algebras [5] was introduced which paved the way for development of BE-algebras [6]. In 2010, B. L. Meng [8] introduced the idea of CI-algebras as a generalization of BE-algebras which is considered to be an important algebraic

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2. Preliminaries

Definition 2.1. *[6]: A non-empty set B, equipped with a binary operation $*$ and a fixed element 1 is said to be a BE-algebra if it satisfies the following postulates:*
(B1) $t * t = 1$,
(B2) $t * 1 = 1$,
(B3) $1 * t = t$,
(B4) $t * (u * v) = u * (t * v)$ *for all* $t, u, v \in B$.

Definition 2.2. *[8]: A non-empty set C, equipped with a binary operation $*$ and a fixed element 1 is said to be a CIâĂŞalgebra if it satisfies the following postulates:*
(C1) $t * t = 1$,
(C2) $1 * t = t$,
(C3) $t * (u * v) = u * (t * v)$ *for all* $t, u, v \in C$.

Example 1. *Let H be a Hilbert space and let $B(H)$ be the class of all bounded linear operators defined on H. Let $C \subset B(H)$ be the set of all positive invertible and commutative operators. We define a binary operation $*$ on C as*

$$P * Q = QP^{-1}, \text{ for all } P, Q \in C.$$

Let I be the identity operator on H. Then $I \in C$. Also for $P, Q, R \in C$, we have
(E1) $P * P = PP^{-1} = I$
(E2) $I * P = PI^{-1} = P$
(E3) $P * (Q * R) = P * (RQ^{-1}) = (RQ^{-1})P - 1 = R(Q^{-1}P^{-1}) = R(P^{-1}Q^{-1})$
$= (RP^{-1})Q^{-1} = Q * (RP^{-1}) = Q * (PR)$
This means that $(C; *, I)$ is a CI-algebra.
A binary relation $\leq$ in C can be defined by $t \leq u$ iff $t * u = 1$.

Definition 2.3. *[8]: A non-empty subset A of a CI-algebra C is said to be a subalgebra of C if $t \in A, u \in A$ imply $t * u \in A$.*

Theorem 2.1. *[11] Let* $(C;*,1)$ *be a CI-algebra and let, the collection of all functions* $h: C \to C$ *be denoted by* $G(C)$*. We define a binary operation* o *in* $G(C)$ *such that for* $h, k \in G(C)$ *and* $t \in C$,

$$(hok)(t) = h(t) * k(t).$$

where $1\tilde{}$ *is defined as* $1\tilde{}(t) = 1$ *for each element* $t \in C$*. Then* $(G(C); o, 1\tilde{})$ *is a CI-algebra. Here two functions* $h, k \in G(C)$ *are equal iff* $h(t) = k(t)$*, for each element* $t \in C$.

Definition 2.4. *Let* $h, k \in G(C)$*. Then composite of* h *and* k*, denoted as* $h \bullet k$*, is defined as*

$$(h \bullet k)(t) = h(k(t)).$$

Definition 2.5. *A multiplier* $h \in G(C)$ *is a mapping such that* $h(t * u) = t * h(u)$ *for all* $t, u \in C$.

3. DUAL MULTIPLIERS IN CI-ALGEBRAS

Definition 3.1. *A dual multiplier* $h \in G(C)$ *is a mapping such that* $h(t * u) = h(t) * u$ *for all* $t, u \in C$.

Note: The identity map $I(t) = t$ is a multiplier as well as a dual multiplier.

Proposition 3.1. *Suppose* h *is a dual multiplier defined on CI-algebra* $(C;*,1)$.
(a) If $h(1) = e$ *then* $h(t) = e * t$ *for any* $t \in C$,
(b) If $h(1) = 1$ *then* h *is the identity map.*

Proof.
(a) Let $h(1) = e$. Since $1 * t = t$ for any $t \in C$, and h is a dual multiplier,

$$h(1 * t) = h(t) \Rightarrow h(1) * t = h(t) \Rightarrow e * t = h(t),$$

(b) Putting $e = 1$ in (a), we get

$$h(t) = 1 * t = t$$

for $t \in C$. So h is the identity map. □

Theorem 3.1. *Composite of two dual multiplier maps is a dual multiplier.*

Proof. Suppose h and k are two dual multiplier maps defined on a CI-algebra $(C;*,1)$. Let $t,u \in C$. Then

$$(h \bullet k)(t*u) = h(k(t*u)) = h(k(t)*u) = h(k(t))*u = (h \bullet k)(t)*u.$$

So $h \bullet k$ is a dual multiplier map. □

As above we can also prove

Corollary 3.1.
(a) If h is multiplier and k is dual multiplier, then

$$(h \bullet k)(t*u) = k(t)*h(u)$$

(b) If h is dual multiplier and k is multiplier, then

$$(h \bullet k)(t*u) = h(t)*k(u).$$

Notation: For $h \in G(C)$, let $B_h = \{t \in C : h(t) = t\}$.

Proposition 3.2. *If h is dual multiplier then $h(1) \neq 1$ iff B_h is empty.*

Proof. Suppose h is a dual multiplier and $h(1) \neq 1$. If possible, suppose B_h is non-empty and $t \in B_h$. Now, we have $t*t = 1$ and so

$$h(1) = h(t*t) = h(t)*t = t*t = 1,$$

which contradicts our assumption that $h(1) \neq 1$. So $B_h = \phi$.

Again, let us assume B_h is empty. Suppose $h(1) = 1$. Then $1 \in B_h$ which is a contradiction to the fact that B_h is empty. Hence $h(1) \neq 1$. □

Proposition 3.3. *If h is a dual multiplier and B_h is non-empty then B_h is a sub-algebra.*

Proof. Suppose h is a dual multiplier and let $m,n \in B_h$. We have $h(m) = m$ and $f(n) = n$. Now $h(m*n) = h(m)*n = m*n \Rightarrow m*n \in B_h$.

Therefore, B_h is a sub-algebra. □

Definition 3.2. *Suppose $(C;*,1)$ is a CI-algebra. We define an addition '$+'$ in C as*

$$t + u = (t*u)*u \text{ for all } t,u \in C.$$

Theorem 3.2. *Suppose h is a dual multiplier on a CI-algebra C. Then*

(i) B_h is closed w. r. t. operation '+';

(ii) $t \in B_h$ and $t \leq u \Rightarrow u \in B_h$.

Proof.
(i) Let $t, u \in B_h$. Then $h(t) = t$ and $h(u) = u$. Now

$$\begin{aligned} h(t+u) = h((t*u)*u) \\ &= (h(t*u))*u \\ &= (h(t)*u)*u \\ &= (t*u)*u \\ &= t+u. \end{aligned}$$

This implies that $t + u \in B_h$ and proves the result.
(ii) Given $t \in B_h$ and $t \leq u \Rightarrow h(t) = t$ and $t * u = 1$. Now

$$\begin{aligned} h(u) = h(1*u) &= h((t*u)*u) \\ &= (h(t*u))*u \\ &= (h(t)*u)*u \\ &= (t*u)*u \\ &= 1*u = u. \end{aligned}$$

This proves that $u \in B_h$. □

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