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κ-ANTI-FUZZY SUBGROUP

B. ANITHA

ABSTRACT. An overview of κ-anti fuzzy subgroup of a group is presented and some related basic results are approached in this paper. In addition, the κ-anti fuzzy subgroup of a group is characterized. Moreover some properties of κ-anti fuzzy subgroup under group homomorphism are investigated.

1. INTRODUCTION

The idea of fuzzy set which is presented by Zadeh [13] have been made use of in the classical mathematics reusage. Fuzzy subgroup of a group was investigated by Rosenfeld [6]. The definition of Fuzzy ideals of a ring was put forth by W. Liu [4]. The ideologies of $(\in, \in \vee q)$-fuzzy groups [5, 6] and $(\in, \in \vee q)$-fuzzy subring [1] was explained clearly by Bhakat and Das. The illustrations of (λ, μ)-fuzzy groups [10] and (λ, μ)-fuzzy subring [11] was presented by B. Yao. The research done by Shen [8] was based on anti-fuzzy subgroups and the study of the product of anti-fuzzy subgroups was done by Dong [3]. While α-fuzzy subgroup got introduced by Sharma [7], the theories of λ-fuzzy subgroup got presented by Sowmya and Sr.Magie Jose [9]. This paper aims at the concept of κ-anti fuzzy subgroup of a group and the characteristics of the same are discussed.

[1]*corresponding author*

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2. Preliminaries

Definition 2.1. *[5] A fuzzy subset (FS) A in a set X is a function $A : X \to [0,1]$.*

Definition 2.2. *If E is a FS of Y, then we denote $E_{(\iota)} = \{\upsilon \in Y | E(\upsilon) < \iota\}$ $\forall \iota \in [0,1]$.*

Definition 2.3. *[1] A FS E of a group G is called a fuzzy subgroup (FSG) if $\forall q, n \in G$,*

(i) $E(qn) \geq \min\{E(q), E(n)\}$
(ii) $E(q^{-1}) \geq E(q)$.

Definition 2.4. *[2] A FS E of a group G is called an anti fuzzy subgroup (AFSG) if $\forall q, n \in G$,*

(i) $E(qn) \leq \max\{E(q), E(n)\}$
(ii) $E(q^{-1}) \leq E(q)$.

Definition 2.5. *[12] A FS E of a group G is called a (λ, μ)-anti-fuzzy subgroup ((λ, μ)AFSG) if $\forall r, n, q \in G$,*

(i) $E(rn) \wedge \mu \leq E(r) \vee E(n) \vee \lambda$
(ii) $E(q^{-1}) \wedge \mu \leq E(q) \vee \lambda$.

Definition 2.6. *[9] Let E be a FS in a groupoid G and $\lambda \in (0,1]$ Then E is called a $\lambda-$ fuzzy subgroupoid ($\lambda-$FSGD) of G if $E(qn) \geq E(q) \wedge E(n) \wedge \lambda$.*

Definition 2.7. *[9] Let E be a FS in a groupoid G and $\lambda \in (0,1]$ Then E is called a λ- fuzzy subgroup ($\lambda-$FSG) of G if*

(i) $E(qn) \geq E(q) \wedge E(n) \wedge \lambda$
(ii) $E(q) \wedge E(q^{-1}) \geq E(q) \wedge \lambda$.

3. κ-Anti-Fuzzy Subgroup

Definition 3.1. *Let P be a FS in a groupoid G and $\kappa \in (0,1]$ Then P is said to be a κ-anti fuzzy subgroupoid (κ-AFSGD) of G if $P(\upsilon\zeta) \leq P(\upsilon) \vee P(\zeta) \vee \kappa$ for every $\upsilon, \zeta \in G$.*

Definition 3.2. *Let P be a FS in a groupoid G and $\kappa \in (0,1]$ Then P is said to be a κ-anti fuzzy subgroup (κ-AFSG) of G if*

(i) $P(\upsilon\zeta) \leq P(\upsilon) \vee P(\zeta) \vee \kappa$
(ii) $P(\upsilon) \vee P(\upsilon^{-1}) \leq P(\upsilon) \vee \kappa,$ *for every* $\upsilon, \zeta \in G.$

Remark 3.1. *Every AFSG is a κ-AFSG, but the converse is not true. It is shown by the following example.*

Example 1. *Let G be the group of integers with respect to addition. Define fuzzy subset P as follows:*

$$P(x) = \begin{cases} 0.7 & \text{if } x \in 6Z \\ 0.5 & \text{if } x \notin 6Z \end{cases}$$

Clearly P is not a AFSG of G since $P(4+2) \nleq max\{P(4), P(2)\}$ as $P(4+2) = 0.7$ whereas $max\{P(4), P(2)\} = 0.5$.

Take $\kappa = 0.7$. Then $P(\upsilon\zeta) \leq P(\upsilon) \vee P(\zeta) \vee 0.7$ and $P(\upsilon) \vee P(\upsilon^{-1}) \leq P(\upsilon) \vee 0.7,$ for every $\upsilon, \zeta \in G$. That is P is a κ-AFSG of G.

Theorem 3.1. *If P is a κ-AFSGD of a finite group G, then P is a κ-AFSG of G.*

Proof. Let $\upsilon \in G$. Since G is finite, it is possible to find an integer $n > 0$ such that $\upsilon^n = e$, where e is an identity in G. Hence $\upsilon^{-1} = \upsilon^{n-1}$. Now,

$$P(\upsilon^{-1}) = P(\upsilon^{n-1}) = P(\upsilon.\upsilon^{n-2}) \leq P(\upsilon) \vee P(\upsilon^{n-2}) \vee \kappa = P(\upsilon) \vee \kappa.$$

That is, $P(\upsilon) \vee P(\upsilon^{-1}) \leq P(\upsilon) \vee P(\upsilon) \vee \kappa = P(\upsilon) \vee \kappa$. Thus P is a κ-AFSG of G. □

Theorem 3.2. *Intersection of two κ-AFSG of G is again a κ-AFSG of G.*

Proof. Let P and Q be two κ-AFSG of G.

$$\begin{aligned}(P \cap Q)(\upsilon\zeta) =& P(\upsilon\zeta) \vee Q(\upsilon\zeta) \leq (P(\upsilon) \vee Q(\zeta) \vee \kappa) \vee (Q(\upsilon) \vee Q(\zeta) \vee \kappa) \\ =& (P(\upsilon) \vee Q(\upsilon) \vee (P(\zeta) \vee Q(\zeta) \vee \kappa) \\ =& (P \cap Q)(\upsilon) \vee (P \cap Q)(\zeta) \vee \kappa.\end{aligned}$$

Also

$$\begin{aligned}(P \cap Q)(\upsilon) \vee (P \cap Q)(\upsilon^{-1}) &= (P(\upsilon) \vee Q(\upsilon)) \vee (P(\upsilon^{-1}) \vee Q(\upsilon^{-1})) \\ &= (P(\upsilon) \vee P(\upsilon^{-1})) \vee (Q(\upsilon) \vee Q(\upsilon^{-1})) \\ &\leq (P(\upsilon) \vee \kappa) \vee (Q(\upsilon) \vee \kappa) \\ &= P(\upsilon) \vee Q(\upsilon) \vee \kappa = (P \cap Q)(\upsilon) \vee \kappa.\end{aligned}$$

Therefore $P \cap Q$ is a κ- AFSG of G. □

Corollary 3.1. *Intersection of a family of κ-AFSG of a G is again a κ-AFSG of G.*

Remark 3.2. *The union of two κ-AFSG not necessarily be a κ-AFSG. The example below depicts the same.*

Example 2. *Let $G = Z$, the set of integers under ordinary addition. Define two FS in G as follows:*

$$P(v) = \begin{cases} 0.4 & \textit{if } v \in 4Z \\ 0.6 & \textit{if } v \notin 4Z \end{cases} \quad \textit{and} \quad Q(v) = \begin{cases} 0.3 & \textit{if } v \in 3Z \\ 0.6 & \textit{if } v \in 3Z \end{cases}.$$

Then P and Q are 0.5- AFSG of G. Now,

$$(P \cup Q)(v) = \begin{cases} 0.4 & \textit{if } v \in 4Z \\ 0.3 & \textit{if } v \in 3Z - 4Z \\ 0.6 & \textit{if } v \notin 4Z, v \notin 3Z \end{cases}.$$

Clearly $P \cup Q$ is not a $0.5-$AFSG of G since $(P \cup Q)(4+9) \nleq (P \cup Q)(4) \vee (P \cup Q)(9) \vee 0.5$ as $(P \cup Q)(4+9) = 0.6$ whereas $(P \cup Q)(4) \vee (P \cup Q)(9) \vee 0.5 = 0.5$. Hence $P \cup Q$ is not a κ-AFSG of G.

Theorem 3.3. *Let P be a κ-AFSG of a group G. Then $P(e) \leq P(v) \vee \kappa, \forall v \in G$.*

Proof. Let v,$v^{-1} \in G$. Then

$$P(e) = P(vv^{-1}) \leq P(v) \vee P(v^{-1}) \vee \kappa \leq P(v) \vee \kappa.$$

$\square$

Theorem 3.4. *If P is a κ-AFSG of a group G, then $P(v\zeta^{-1}) = P(e) \Rightarrow P(v) \leq P(\zeta) \vee \kappa \; \forall v \in G$.*

Proof. If P is a κ-AFSG of a group G, for all $v, \zeta \in G$, $P(v) = P(v\zeta^{-1}\zeta) \leq P(v\zeta^{-1}) \vee P(\zeta) \vee \kappa = P(e) \vee P(\zeta) \vee \kappa = P(\zeta) \vee \kappa$. $\square$

Theorem 3.5. *Let P be FS of a group G and let $q = \sup\{P(v) : v \in G\} \neq 0$. Then P is a κ-AFSG of G $\forall \kappa \geq q$.*

Proof. Let $\kappa \geq q = \sup\{P(v) : v \in G\}$. $P(v) \leq \kappa, \forall v \in G$. Therefore $P(v\zeta) \leq \kappa$ and $P(v) \vee P(\zeta) \vee \kappa = \kappa$. That is $P(v\zeta) \leq P(v) \vee P(\zeta) \vee \kappa$.

Also since, $P(v^{-1}) \vee \kappa = \kappa$. $P(v) \vee P(v^{-1}) \leq P(v^{-1}) \vee \kappa = \kappa$. Hence P is a $\kappa-$AFSG of G. $\square$

Theorem 3.6. *Let $\iota \in (0,1]$, $P(e) \leq \iota$ and P be a $\kappa \in [0,\iota]$-AFSG of a Group G. Then the level set $P_{(\iota)}$ is a subgroup (SG) of G.*

Proof. Clearly P is non empty. Let $\upsilon, \zeta \in P_{(\iota)}$ then $P(\upsilon) \leq \iota$, $P(\zeta) \leq \iota$. Since P is a κ-AFSG of G, $P(\upsilon\zeta) \leq P(\upsilon) \vee P(\zeta) \vee \kappa \leq \iota$. Hence $\upsilon\zeta \in P_{(\iota)}$.

Also $\upsilon \in P_{(\iota)}$ implies $P(\upsilon) \leq \iota$. Since P is a κ-AFSG of G, $P(\upsilon) \vee P(\upsilon^{-1}) \leq P(\upsilon) \vee \kappa \leq \iota$. It shows that $P(\upsilon^{-1}) \leq \iota$. This means $\upsilon^{-1} \in P_{(\iota)}$. Therefore $P_{(\iota)}$ is a SG of G. □

Theorem 3.7. *Let P be a FS of a group G such that $P_{(\iota)}$ is a SG of G. $\forall \iota \in [0,1]$, $P(e) \leq \iota$, then P is a $\kappa = \vartheta-$AFSG of G, where $\vartheta = inf\{P(\upsilon) : \upsilon \in G\}$.*

Proof. Let $\upsilon, \zeta \in G$ and let $P(\upsilon) = \iota_1$ and $P(\zeta) = \iota_2$. Then $\upsilon \in P_{(\iota_1)}$, $\zeta \in P_{(\iota_2)}$. Let us assume that $\iota_1 \geq \iota_2$. Then $P_{(\iota_2)} \subseteq P_{(\iota_1)}$. Thus $\upsilon, \zeta \in P_{(\iota_1)}$ and since $P_{(\iota_1)}$ is a subgroup of G, $\upsilon\zeta \in P_{(\iota_1)}$. Therefore $P(\upsilon\zeta) \leq \iota_1 = P(\upsilon) \vee P(\zeta) \vee \kappa$, where $\kappa = \vartheta$. Also let $\upsilon \in G$ and $P(\upsilon) = \iota$. Then $\upsilon \in P_{(\iota)}$. Since $P_{(\iota)}$ is SG of G, $\upsilon^{-1} \in P_{(\iota)}$. Which implies that $P(\upsilon^{-1}) \leq \iota$. Now $P(\upsilon) \vee P(\upsilon^{-1}) \leq P(\upsilon) \vee \kappa = \iota$, where $\kappa = \vartheta$. Thus P is a $\kappa = \vartheta-$AFSG of G. □

Theorem 3.8. *A FS P in a group G is a κ-AFSG of G if and only if $P(\upsilon\zeta^{-1}) \leq P(\upsilon) \vee P(\zeta) \vee \kappa$.*

Proof. Suppose P is a κ-AFSG of G. Then

$$\begin{aligned}
P(\upsilon\zeta^{-1}) &= P(\upsilon\zeta^{-1}\zeta\zeta^{-1}) \leq P(\upsilon\zeta^{-1}) \vee P(\zeta\zeta^{-1}) \vee \kappa \\
&\leq P(\upsilon\zeta^{-1}) \vee P(\zeta) \vee P(\zeta^{-1}) \vee \kappa \\
&\leq P(\upsilon\zeta^{-1}) \vee P(\zeta) \vee \kappa \leq P(\upsilon) \vee P(\zeta^{-1}) \vee P(\zeta) \vee \kappa \\
&\leq P(\upsilon) \vee P(\zeta) \vee \kappa.
\end{aligned}$$

Conversely, let $P(\upsilon\zeta^{-1}) \leq P(\upsilon) \vee P(\zeta) \vee \kappa$ for a FS Pin G,

$$P(\zeta^{-1}) = P(e\zeta^{-1}) \leq P(e) \vee P(\zeta) \vee \kappa \leq P(\zeta) \vee \kappa\,.$$

Hence

$$P(\zeta^{-1}) \vee P(\zeta) \leq P(\zeta) \vee P(\zeta) \vee \kappa = P(\zeta) \vee \kappa.$$

Then

$$P(\upsilon\zeta) = P(\upsilon(\zeta^{-1})^{-1}) \leq P(\upsilon) \vee P(\zeta^{-1}) \vee \kappa.$$

So $P(\upsilon\zeta) \vee P(\zeta) \leq P(\upsilon) \vee P(\zeta^{-1}) \vee \kappa \vee P(\zeta) \leq P(\upsilon) \vee P(\zeta) \vee \kappa$ (since $P(\zeta^{-1}) \vee P(\zeta) \leq P(\zeta) \vee \kappa$).

But

$$P(\upsilon\zeta) \leq P(\upsilon\zeta) \vee P(\zeta) \leq P(\upsilon) \vee P(\zeta) \vee \kappa\,.$$

Hence P is a κ- AFSG of G. □

Theorem 3.9. *Let P be the characteristic function of a non-empty subset H of a group G. Then P is a κ-AFSG of G if and only if H is a SG of G.*

Proof. Clearly P is a FS in G. First let P be a κ-AFSG of G. For $\upsilon, \zeta \in H$, $P(\upsilon) = 1$ and $P(\zeta) = 1$. Now,

$$P(\upsilon\zeta) \leq P(\upsilon) \vee P(\zeta) \vee \kappa = 1 \vee 1 \vee \kappa \Rightarrow P(\upsilon\zeta) = 1.$$

Thus $\upsilon\zeta \in H$. Also $P(\upsilon) \vee P(\upsilon^{-1}) \leq P(\upsilon) \vee \kappa = 1 \Rightarrow P(\upsilon^{-1}) = 1 \Rightarrow \upsilon^{-1} \in H$. Therefore H is a SG of G.

Conversely, if H is a SG of G then its characteristic function is FSG of G and hence is a κ-AFSG of G. □

4. Homomorphism and κ-Anti-Fuzzy Subgroup

Theorem 4.1. *A homomorphic preimage of a κ-AFSG E of a group G_2 is a κ-AFSG of group G_1 where $\eta^{-1}(E)(\upsilon) = E(\eta(\upsilon)); \forall \upsilon \in G_1$.*

Proof. Let $\eta : G_1 \to G_2$ be a group homomorphism. Let E be a κ-AFSG of a group G_2. For $\upsilon, \zeta \in G_1$,

$$\begin{aligned}\eta^{-1}(E)(\upsilon\zeta) &= E[\eta(\upsilon\zeta)] = E[\eta(\upsilon)\eta(\zeta)] \leq E[\eta(\upsilon)] \vee E[\eta(\zeta)] \vee \kappa \\ &= \eta^{-1}(E)(\upsilon) \vee \eta^{-1}(E)(\zeta) \vee \kappa.\end{aligned}$$

Also

$$\begin{aligned}\eta^{-1}(E)(\upsilon) \vee \eta^{-1}(E)(\upsilon^{-1}) &= E[\eta(\upsilon)] \vee E[\eta(\upsilon^{-1})] = E[\eta(\upsilon)] \vee E[\eta(\upsilon)^{-1}] \\ &\leq E[\eta(\upsilon)] \vee \kappa = \eta^{-1}(E)(\upsilon) \vee \kappa.\end{aligned}$$

Thus $\eta^{-1}(E)$ is a κ-AFSG of a group G_1. □

Theorem 4.2. *Let P be a κ-AFSG of a group G_1. If $\eta : G_1 \to G_2$ is a bijective homomorphism, then $\eta(P)$ is a κ-AFSG of a group G_2 where*

$$\eta(P)(\zeta) = \inf_{\upsilon \in G_1} \{P(\upsilon) | \eta(\upsilon) = \zeta\},$$

for all $\zeta \in G_2$.

Proof. Let $\zeta_1, \zeta_2 \in G_2$, we have,

$$\text{(i) } \eta(P)(\zeta_1\zeta_2) = \inf\{P(\upsilon_1\upsilon_2)|\eta(\upsilon_1\upsilon_2) = \zeta_1\zeta_2\}$$
$$\leq \inf\{P(\upsilon_1) \vee P(\upsilon_2) \vee \kappa|\eta(\upsilon_1) = \zeta_1, \eta(\upsilon_2) = \zeta_2\}$$
$$= \inf\{P(\upsilon_1)|\eta(\upsilon_1) = \zeta_1\} \vee \inf\{P(\upsilon_2)|\eta(\upsilon_2) = \zeta_2\} \vee \kappa$$
$$= \eta(P)(\zeta_1) \vee \eta(P)(\zeta_2) \vee \kappa.$$
$$\text{(ii) } \eta(P)(\zeta) \vee \eta(P)(\zeta^{-1}) = \inf\{P(\upsilon)|\eta(\upsilon) = \zeta\} \vee \inf\{P(\upsilon^{-1})|\eta(\upsilon^{-1}) = \zeta^{-1}\}$$
$$= \inf\{P(\upsilon) \vee P(\upsilon^{-1})|\eta(\upsilon) = \zeta, \eta(\upsilon^{-1}) = \zeta^{-1}\}$$
$$\leq \inf\{P(\upsilon) \vee \kappa|\eta(\upsilon) = \zeta\} = \inf\{P(\upsilon) \vee |\eta(\upsilon) = \zeta\} \vee \kappa$$
$$= \eta(P)(\zeta) \vee \kappa.$$

Thus $\eta(P)$ is a κ-AFSG of a group G_2. $\square$

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DEPARTMENT OF MATHEMATICS
GOVERNMENT ARTS COLLEGE, CHIDAMBARAM-608102
TAMIL NADU, INDIA