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A NUMERICAL METHOD FOR FRACTIONAL VARIATIONS PROBLEMS BASED ON FRACTIONAL ORDER EULER FUNCTIONS

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ABSTRACT. In this paper, we present a numerical method based on Fractional order Euler functions (FEFs) to obtain the solution of a class of fractional variational problems. We construct the operational matrix for fractional differentiation of these functions and a numerical method is developed by applying the properties of FEFs along with the operational matrix. Numerical examples including comparison with other methods are illustrated to express the efficiency and simplicity of the proposed method.

1. INTRODUCTION

Fractional calculus, which deals with the study of arbitrary order integrals and derivatives, has found several applications over the years in many distinct fields. It has been applied to model numerous real world problems in physics, signal and image processing, mechanics and dynamic systems, biology, environmental science, materials, economic, multidisciplinary in engineering fields etc., [1]. In [2–5] some numerical methods were presented to obtain approximate solution for problems in fractional calculus. It has been done in the field of fractional calculus of variations as well [6, 7].

The theory of the fractional calculus of variations was introduced by Riewe in 1996, to deal with nonconservative systems in mechanics [8]. It is a problem in

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which either the objective functional or the constraint equations or both contain at least one fractional derivative term. Recently, special kinds of polynomials or functions such as Legendre orthonormal polynomials [9], Jacobi orthonormal polynomials [10], Haar wavelets [12] was implemented to obtain numerical solution of these problems. In the present paper, Fractional order Euler polynomials (FEPs) will be applied for solving a class of fractional variational problems.

This paper is structured as follows: The basic definitions and notations that we have used, are given in section 2. Section 3 describes the generating formula and properties of fractional order Euler functions. Section 4 is devoted to derive the operational matrix for fractional differentiation of FEFs. In Section 5, we present our proposed method for the solution of fractional variational problems. In the last section some numerical examples along with the comparison with other methods are discussed.

2. Preliminaries

Definition 2.1. *[15] The Caputo fractional derivative of order $\alpha > 0$ is defined as follows:*

$$ {}_0^C\mathcal{D}_x^\alpha f(x) = \begin{cases} \frac{1}{(m-\alpha)} \int_0^x \frac{f^{m(t)}}{(x-t)^{\alpha+1-m}} dt, & m-1 < \alpha < m \\ \\ \frac{d^m}{dx^m} f(x), & \alpha = m \end{cases} .$$

Some of the properties of the operators $\mathcal{I}_x^\alpha$ and ${}_0^C\mathcal{D}_x^\alpha$ are given below [15]:

(1) ${}_0^C\mathcal{D}_x^\alpha(\mathcal{I}_x^\alpha f(x)) = f(x)$
(2) $\mathcal{I}_x^\alpha({}_0^C\mathcal{D}_x^\alpha f(x)) = f(x) - \sum_{k=0}^{n-1} f^k(0^+)\frac{x^k}{k}$
(3) ${}_0^C\mathcal{D}_x^\alpha(C) = 0$
(4) ${}_0^C\mathcal{D}_x^\alpha(x^p) = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}, & p \in \mathbb{R}, p > n-1, n-1 < \alpha < n \\ 0, & p \in \mathbb{N}, p > n-1, n-1 < \alpha < n \end{cases}$
(5) ${}_0^C\mathcal{D}_x^\alpha(\lambda f(x) + g(x)) = \lambda {}_0^C\mathcal{D}_x^\alpha f(x) + {}_0^C\mathcal{D}_x^\alpha g(x)$.

3. Fractional order Euler functions

In this section we recall definition and some properties of fractional order Euler functions [14]. The analytic form of $E_m^\beta(x)$ is given by

$$\sum_{k=0}^{m} E_k^\beta(x) + E_m^\beta(x) = 2x^{m\beta}.$$

Properties:

(1) Let $E^\beta(x) = [E_0^\beta(x), E_1^\beta(x), \ldots, E_m^\beta(x)]^T$ and $X^\beta = [1, x^\beta, \ldots, x^{m\beta}]^T$, then fractional order Euler functions can be represented in a matrix form as

$$E^\beta(x) = BX^\beta$$

where,

$$B = \begin{bmatrix} \frac{2-2^2}{1}\begin{pmatrix}1\\1\end{pmatrix}B_1(0) & 0 & \ldots & 0 \\ \frac{2-2^3}{2}\begin{pmatrix}2\\2\end{pmatrix}B_2(0) & \frac{2-2^2}{2}\begin{pmatrix}2\\1\end{pmatrix}B_1(0) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2-2^{m+2}}{m+1}\begin{pmatrix}m+1\\m+1\end{pmatrix}B_{m+1}(0) & \frac{2-2^{m+1}}{m+1}\begin{pmatrix}m+1\\m\end{pmatrix}B_m(0) & \ldots & \frac{2-2^2}{m+1}\begin{pmatrix}m+1\\1\end{pmatrix}B_1(0) \end{bmatrix}$$

(2) $$\int_0^1 E_m^\beta(x)E_n^\alpha(x)x^{1-\beta}dx = (-a)^{n-1}\frac{m(n+1)}{(m+n+1)}E_{m+n+1}(0)\ m,n \geqslant 1.$$

(3) **Approximation of Functions:** A function $f(x)$, square integrable in [0,1], can be expanded as $f(x) = \sum_{i=0}^{\infty} c_i E_i^\beta(x)$. To evaluate C, we solve $C^T = A^T D^{-1}$ where $A^T = [a_0, a_1, a, ..., a_m]$, $\quad a_j = \int_0^1 E_j^\beta(x)f(x)x^{1-\beta}$ and $D = [d_{ij}^\beta]$, $\quad d_{ij} = \displaystyle\int_0^1 E_i^\beta(x)E_j^\beta(x)x^{\beta-1}dx$.

(4) **Error Analysis:** The error analysis is given by the following theorem.

Theorem 3.1. *Suppose that $D^{k\alpha}f(x) \in C[0,1]$ for $k = 1,2,...m$ and $Y_m^\alpha = span\{E_0^\alpha(x), E_1^\alpha(x), ...E_m^\alpha(x)\}$ is a vector space. If $f_m(x)$ is the best approximation of f out of Y_m^α, then the mean error bound is presented as follows:*

$$\|f - f_m\| \leqslant \frac{M_\alpha}{\Gamma((m+1)\alpha+1)\sqrt{(2m+2)\alpha+1}},$$

where, $M_\alpha \geqslant \sup_{\xi\in[0,1]} |{}_0^C\mathcal{D}_{a^+}^{(m+1)\alpha} f(x)|$.

4. Operational matrix for Fractional Differentiation for FEFs

Let M^α be the $(m+1)\times(m+1)$ Caputo fractional operational matrix of differentiation of order α for FEFs of order $m\alpha$. The Caputo fractional differentiation of the vector $E^\alpha(x)$ can be expressed by

$$ {}_0^C\mathcal{D}_x^\alpha E^\alpha(x) = M^\alpha E^\alpha(x). $$

Now by using the matrix representation of FEFs and the properties of Caputo derivatives, we have

$$ \begin{aligned} {}_0^C\mathcal{D}_x^\alpha E^\alpha(x) &= {}_0^C\mathcal{D}_x^\alpha B X^\alpha = B\,{}_0^C\mathcal{D}_x^\alpha X^\alpha = BN^\alpha X^\alpha \\ &= BN^\alpha(B^{-1}E^\alpha(x)) = (BN^\alpha B^{-1})E^\alpha(x), \end{aligned} $$

where N^α is the operation matrix of Caputo fractional differentiation of the vector $X^\alpha = [1, x^\alpha, x^{2\alpha}, ...x^{m\alpha}]^T$ and it is given by

$$ N^\alpha = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \frac{\Gamma(\alpha+1)}{\Gamma(0\alpha+1)} & 0 & \cdots & 0 & 0 \\ 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(m\alpha+1)}{\Gamma((m-1)\alpha+1)} & 0 \end{pmatrix}. $$

Then the Caputo fractional differentiation operational matrix for FEFs will be $M^\alpha = (BN^\alpha B^{-1})$.

5. Proposed Method

Consider the problem of extremization of a functional J of the form

$$ J[y(x)] = \int_0^1 F[x, y(x), {}_0^C\mathcal{D}_x^\alpha y(x)]dx, \qquad 0 < \alpha \le 1 \tag{5.1} $$

with

$$ y(0) = y_0, y(1) = y_1, $$

where ${}_0^C\mathcal{D}_x^\alpha$ is the fractional order derivative in the Caputo sense. If $y(1)$ is unspecified, we consider

$$ \left[\frac{\partial F}{\partial\,{}_0^C\mathcal{D}_x^\alpha y}\right]_{x=1} = 0. $$

This problem can be solve by using the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} +_x^C \mathcal{D}_1^\alpha \frac{\partial F}{\partial_0^C \mathcal{D}_x^\alpha y} = 0.$$

In our proposed method, We approximate $y(x)$ in terms of FEFs as follows:

$$y(x) \approx C^T E^\alpha(x),$$

where $C^T = \begin{bmatrix} c_0 & c_1 & \dots & c_m \end{bmatrix}$ and $E^\alpha(x) = \begin{bmatrix} E_0^\alpha(x) & E_1^\alpha(x) & \dots & E_m^\alpha(x) \end{bmatrix}$. Then

$${}_0^C\mathcal{D}_x^\alpha y(x) = ({}_0^C\mathcal{D}_x^\alpha)(C^T E^\alpha(x)) = C^T({}_0^C\mathcal{D}_x^\alpha E^\alpha(x)) = C^T(M^\alpha E^\alpha(x))$$

The other terms in the functional of equation (5.1) can also be expressed in terms of FEFs through FEFs approximation then J becomes

$$J = J[c_0, c_1, ...c_m]. \tag{5.2}$$

i.e., the original fractional variational problem of extremization (5.1) becomes extremization of a finite set of variables in (5.2). Taking partial derivatives of J with respect to c_i and setting them equal to zero, we get,

$$\frac{\partial J}{\partial c_i} = 0, \qquad i = 0, 1, ..., m.$$

Solving for c_i by using equation (5) and the conditions

$$[C^T E(x)]_{x=0} = y_0$$

and

$$\left[\frac{\partial F}{\partial_0^C \mathcal{D}_x^\alpha y}\right]_{x=1} = 0.$$

Substitutting all c_is in equation (5), we get the desired solution.

6. Numerical Examples

Example 1. *Consider the problem of extremization of the functional*

$$J[y(x)] = \int_0^1 \frac{1}{2}({}_0^C\mathcal{D}_x^\alpha y(x) - 1)^2 dx, \qquad 0 < \alpha \leq 1$$

with $y(0) = 1 - \frac{1}{\Gamma(\alpha+1)}$ *and* $y(1) = 1$.

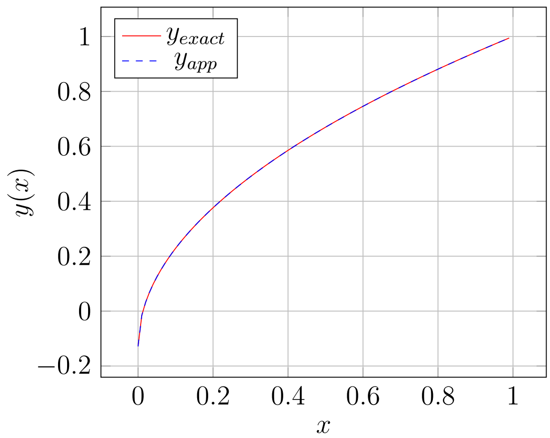
For $0 < \alpha \leq 1$*, The exact solution of this problem is given by*

$$y_{exact}(x) = 1 - \frac{1}{\Gamma(\alpha+1)} + \frac{x^\alpha}{\Gamma(\alpha+1)}.$$

Table 1 give the approximate solution of the above example for $\alpha = 0.5$ *and* $\alpha = 0.75$ *with different values of m and compares the results for with the exact solution. For* $\alpha = 1$ *we get the approximate solution same as the exact solution of this problem 2. ie.,* $y_{app} = x = y_{exact}$

x	$(\alpha = 0.5)$				Error $(\alpha = 0.75)$				
	$y_{app}(m = 2)$	$y_{app}(m = 3)$	y_{exact}	absolute error	$y_{app}(m = 2)$	$y_{app}(m = 3)$	$y_{app}(m = 4)$	y_{excact}	absolute error
0	-0.12837917	-0.12837917	-1.2837917	1.387779×10^{-16}	-0.088068	-0.088068	-0.088068	-0.088065252	2.748×10^{-6}
0.125	0.27056341	0.27056341	0.27056311	2.944762×10^{-7}	0.14067039	0.14067045	0.14067106	0.14067229	1.228647×10^{-6}
0.250	0.43581083	0.43581083	0.43581042	4.164522×10^{-7}	0.29662263	0.29662266	0.29662237	0.29662391	1.53673×10^{-6}
0.375	0.56260964	0.56260964	0.56260913	5.100477×10^{-7}	0.43334289	0.43334287	0.43334225	0.43334359	1.341347×10^{-6}
0.5	0.66950598	0.66950598	0.66950539	5.889524×10^{-7}	0.55890207	0.55890201	0.55890175	0.55890222	4.646998×10^{-6}
0.625	0.76368355	0.76368355	0.76368289	6.584688×10^{-7}	0.67676521	0.67676512	0.67676565	0.67676482	8.306047×10^{-6}
0.750	0.84882658	0.84882658	0.84882586	7.213164×10^{-7}	0.78883734	0.78883724	0.78883848	0.7888364	2.077168×10^{-6}
0.875	0.92712367	0.92712367	0.92712289	7.791108×10^{-7}	0.89631213	0.89631207	0.89631336	0.89631067	2.688511×10^{-6}
1	1.0000008	1.0000008	1	8.329044×10^{-7}	1.000002	1.000002	1.000002	1	2.000048×10^{-6}

TABLE 1. Approximate solution y_{app} of the example 2 for $\alpha = 0.5$ and $\alpha = 0.75$ with different values of m.



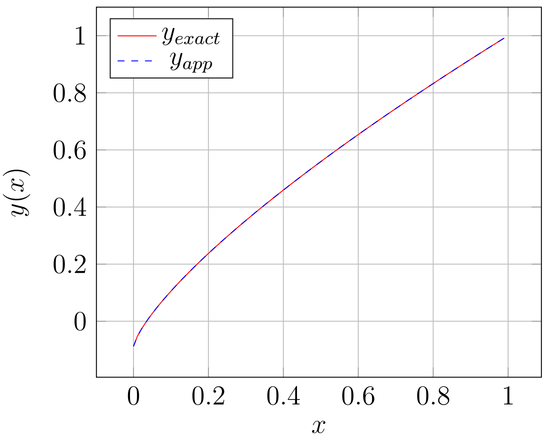


FIGURE 1. **(a)**:The approximate solution by FEFs with m=3 and exact solution of Example 2 for $\alpha = 0.5$.**(b)**:The approximate solution by FEFs with m=4 and exact solution of Example 2 for $\alpha = 0.75$.

7. CONCLUSION

In this article, an operational matrix for fractional differentiation of fractional order Euler functions is constructed. The fractional derivatives are described in the Caputo sense. By using the operational matrix and FEFs, the fractional variational problem is reduced to a system of linear algebraic equations. The numerical examples show that the approximate solution that we get from the suggested method even with four FEFs is more closer to the exact solution than that from other methods [11, 12] where eight polynomials were taken.

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