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# INTERVAL-VALUED FUZZY IDEALS IN ORDERED SEMIRINGS

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ABSTRACT. In this article the conception of interval-valued (i-v) fuzzy ideals(IVFIs) in ordered semirings (OSs) are instigated and it is manifested that the set of all IVFIs of an OS forms a zero sum free semiring.

## 1. INTRODUCTION

Semiring is an algebraic structure which is a typical speculation of rings and distributive lattice. As of late, much intrigue is appeared to sum up mathematical structures of groups, rings, semi-rings, near-rings. Fuzzy set theory has extraordinary lavishness in applications than the conventional set theory. Following the disclosure of fuzzy sets, much consideration has been paid to sum up the fundamental ideas of classical algebra in a fuzzy system and consequently the rise of fuzzy algebra is unavoidable. Ordered and interior ideals of an ordered semiring have been concentrated in [6] and its fuzzy idea has been investigated in [3]. i-v fuzzy subsets were presented by zadeh [16, 17] as an expansion of fuzzy sets. Thus fuzzy subgroup was talked about by Biswas [2]. For chips away at fuzzy semirings see [1, 4, 5, 7].

For more works on i-v fuzzy ideals and i-v intuitionistic fuzzy near- rings see, [8–15].

In section 2, we present IVFIs in OS and study a portion of their related properties. In section 3, we characterize intersection, composition and addition so as

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to examine the structure of i-v fuzzy ideals OS. It is uncovered that the set of all IVFIs of an OS forms a zero sum free semiring under addition and composition of IVFI.

## 2. Preliminaries

For fundamental meanings of semirings and ideals see [6].

For the itemized definition and different outcomes on i-v fuzzy subset (IVFSS) if it's not too much trouble, see [17].

## 3. Structure of Interval-valued Fuzzy Ideals in Ordered Semirings

All through this paper, except if in any case referenced, $S$ mean OS and $L_S$ signify its characteristic function. Let $\mathbb{A}$ signifies the arrangement of all IVFSSs of $S$.

**Definition 3.1.** *For $\overline{\mu}, \overline{\nu} \in \mathbb{A}$ and $x, y, z \in S$. Define composition and sum of $\overline{\mu}$ and $\overline{\nu}$ as follows:*

$$\begin{aligned} \overline{\mu} \circ_{\overline{\mathrm{I}}} \overline{\nu}(x) &= \sup\{\min\{\overline{\mu}(y), \overline{\nu}(z)\}\} \\ &= 0, \text{ if } x \text{ cannot be expressed as } x \leq yz \end{aligned}$$

*and*

$$\begin{aligned} \overline{\mu} +_{\overline{\mathrm{I}}} \overline{\nu}(x) &= \sup\{\min\{\overline{\mu}(y), \overline{\nu}(z)\}\} \\ &= 0, \ if \ x \not\leq y + z. \end{aligned}$$

**Proposition 3.1.** *If $\overline{\mu}$ is an IVFSS of $S$, then $(L_S \circ_{\overline{\mathrm{I}}} \overline{\mu})(x) \geq (L_S \circ_{\overline{\mathrm{I}}} \overline{\mu})(y)( \ resp.$ $(L_S +_{\overline{\mathrm{I}}} \overline{\mu})(x) \geq (L_S +_{\overline{\mathrm{I}}} \overline{\mu})(y)) \forall x, y \in S$ with $x \leq y$.*

*Proof.* Let $x, y \in S$ with $x \leq y$. If $y \not\leq b_1.b_2$ for $b_1, b_2 \in S$, then nothing to prove. If $y \leq b_1.b_2$, then

$$(L_S \circ_{\overline{\mathrm{I}}} \overline{\mu})(y) = \sup_{y \leq b_1 b_2} \{\min\{L_S(b_1), \overline{\mu}(b_2)\}\} = \sup_{y \leq b_1 b_2} \{\overline{\mu}(b_2)\}.$$

Since $x \leq y \leq b_1 b_2$,

$$(L_S \circ_{\overline{\text{I}}} \overline{\mu})(x) = \sup_{x \leq a_1 a_2} \{\min\{L_S(a_1), \overline{\mu}(a_2)\}\}$$
$$\geq \sup_{x \leq b_1 b_2} \{\min\{L_S(b_1), \overline{\mu}(b_2)\}\}$$
$$= \sup_{x \leq b_1 b_2} \{\overline{\mu}(b_2)\} = (L_S \circ_{\overline{\text{I}}} \overline{\mu})(y).$$

Similarly when $x \leq y$, $(L_S +_{\overline{\text{I}}} \overline{\mu})(x) \geq (L_S +_{\overline{\text{I}}} \overline{\mu})(y)$. □

**Definition 3.2.** *Let $\overline{\mu} \neq \emptyset \in \mathbb{A}$. $\overline{\mu}$ is an i-v fuzzy left ordered ideal (IVFLOI) [resp. i-v fuzzy right ordered ideal(IVFROI)] of $S$ if*

*$x \leq y$ implies $\overline{\mu}(x) \geq \overline{\mu}(y)$ in addition to the definition of IVFIs for $x, y \in S$.*

*By an i-v fuzzy ordered ideal(IVFOI) we mean, it is both an IVFLOI as well as an IVFROI.*

**Example 1.** *Let $S = \{0, l, m, n\}$ with the ordered relation $0 < l < m < n$. Define operations on $S$ by following:*

| ⊕ | 0 | l | m | n |
|---|---|---|---|---|
| 0 | 0 | l | m | n |
| l | l | l | m | n |
| m | m | m | m | n |
| n | n | n | n | n |

and

| ⊙ | 0 | l | m | n |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| l | 0 | l | l | l |
| m | 0 | l | l | l |
| n | 0 | l | l | l |

*Then $(S, \oplus, \odot)$ forms an ordered semiring. Now if we define an i-v fuzzy subset $\overline{\mu}$ of $S$ by $\overline{\mu}(0) = \overline{1}, \overline{\mu}(l) = \overline{0.7}, \overline{\mu}(m) = \overline{0.4}$ and $\overline{\mu}(n) = \overline{0.2}$, then $\overline{\mu}$ will be an IVFOI $S$.*

**Example 2.** *Let $S = \{0, 1, 2, 3\}$ with*

| ∗ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 1 | 3 | 3 |
| 2 | 2 | 3 | 2 | 2 |
| 3 | 3 | 3 | 2 | 2 |

and

| · | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 |
| 2 | 0 | 1 | 2 | 2 |
| 3 | 0 | 1 | 3 | 3 |

*Then $(S, *, \cdot)$ forms an OS with the relation defined by "$x \leq y$" if and only if $x * y * x \cdot y = y, x \cdot y = x$ and $x \neq y$.*

*Let $\overline{\mu} \in \mathbb{A}$ defined by $\overline{\mu}(0) = \overline{1}, \overline{\mu}(1) = \overline{0.9}, \overline{\mu}(2) = \overline{0.4}$ and $\overline{\mu}(3) = \overline{0.6}$. Then $\overline{\mu}$ is not an IVFOI of $S$ as $\overline{\mu}(3 * 3) = \overline{\mu}(2) = \overline{0.4} \not\geq \overline{0.6} = \min\{\overline{\mu}(3), \overline{\mu}(3)\}$.*

**Theorem 3.1.** *An IVFSS $\overline{\mu}$ of $S$ is an IVFOI if and only if its level subset(LS) $\overline{\mu}_{\overline{t}}$ is an ordered ideal(OI) of $S$.*

As an outcome of it, we can acquire the accompanying:

**Theorem 3.2.** *$I(\neq \emptyset \subseteq S)$ is a LOI of $S$ if and only if $L_I$ is an IVFLOI of $S$.*

**Definition 3.3.** *$\overline{\mu}$ be an IVFSS $S$ and $a \in S$. We mean $I_a$ the subset of $S$ characterized as follows:*

$$I_a = \{b \in S | \overline{\mu}(b) \geq \overline{\mu}(a)\}.$$

**Proposition 3.2.** *For an IVFROI(resp. IVFLOI) $\overline{\mu}$ of $S$, $I_a$ is a right (resp. left) ideal (RI or LI)of $S$ for $a \in S$.*

The opposite of the above suggestion is absurd which can be seen by the accompanying model.

**Example 3.** *Let $S = \{0, l, m, n\}$ with the ordered relation $0 < n < m < l$. Define operations on $S$ by following:*

| $\oplus$ | 0 | l | m | n |
|---|---|---|---|---|
| 0 | 0 | l | m | n |
| l | l | l | l | l |
| m | m | l | l | l |
| n | n | n | l | l |

and

| $\odot$ | 0 | l | m | n |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| l | 0 | l | l | l |
| m | 0 | m | m | m |
| n | 0 | n | n | n |

*Then $(S, \oplus, \odot)$ forms an OS.*

*Now suppose $\overline{\mu}$ be an IVFSS of $S$ defined by $\overline{\mu}(0) = \overline{1}, \overline{\mu}(n) = \overline{0.3}, \overline{\mu}(m) = \overline{0.2}$ and $\overline{\mu}(l) = 0.1$. Then $I_0 = \{0\}, I_n = \{0, n\}, I_m = \{0, n, m\}$ and $I_l = \{0, l, m, n\}$- all are RI of $S$. But $\overline{\mu}$ is not an IVFROI, since $\overline{\mu}(m \oplus n) = \overline{\mu}(l) = \overline{0.1} \not\geq \overline{0.2} = \min\{\overline{\mu}(m), \overline{\mu}(n)\}$.*

Denote the set all IVFROIs, IVFLOIs and IVFOIs by $\mathbb{B}, \mathbb{C}$ and $\mathbb{D}$ respectively.

**Proposition 3.3.** *Intersection of a non-empty collection of members of $\mathbb{B}$ (resp. $\mathbb{C}$) is an IVFROI (resp. IVFLOI) of $S$.*

*Proof.* Similarly, the result holds for IVFLOI. □

**Proposition 3.4.** *Let $g : R \to S$ be a morpism of ordered semirings, i.e., semiring homomorpism satisfying additional condition $l \leq m \Rightarrow g(l) \leq g(m)$. Then if $L$ is an IVFLOI of $S$, then $g^{-1}(L)$ is an IVFLOI of $R$.*

*Proof.* Let $g : R \to S$ be a morpism of OSs and $L$ is an IVFLOI of $S$. Now $f^{-1}(L)(0_R) = L(0_S) \geq L(x') \neq 0$ for a $x' \in S$. Therefore $g^{-1}(L) \neq \emptyset$. Now for any $u, v \in R$,

$$\begin{aligned} g^{-1}(L)(u+v) &= L(g(u+v)) \\ &= L(g(u)+g(v)) \\ &\geq \min\{L(g(u)), L(g(v))\} \\ &= \min\{(g^{-1}(L))(u), (g^{-1}(L))(v)\}. \end{aligned}$$

Again $(g^{-1}(L))(uv) = L(g(uv)) = L(g(u)g(v)) \geq L(g(v)) = (g^{-1}(L))(v)$. Also if $u \leq v$, then $g(u) \leq g(v)$. Then

$$(g^{-1}(L))(u) = L(g(u)) \geq L(g(v)) = (g^{-1}(L))(v).$$

Thus $g^{-1}(L)$ is an IVFLOI of $R$. □

Cartesian product of two elements in $\mathbb{C}$ is IVFLOI.

**Theorem 3.3.** *$\overline{\mu}$ and $\overline{\nu} \in \mathbb{C}$ then $\overline{\mu} \times \overline{\nu}$ is an IVFLOI of $S \times S$.*

**Theorem 3.4.** *$\overline{\delta}$ be an IVFSS of $S$. Then $\overline{\delta}$ is an IVFLOI of $S$ if and only if $\overline{\delta} \times \overline{\delta}$ is an IVFLOI of $S \times S$.*

*Proof.* $\overline{\delta}$ be an IVFLOI of $S$. By Theorem 3.15, $\overline{\delta} \times \overline{\delta}$ is an IVFLOI of $S \times S$. Suppose $\overline{\delta} \times \overline{\delta}$ is an IVFLOI of $S \times S$. Let $a_1, a_2, b_1, b_2 \in S$. Then

$$\begin{aligned} \min\{\overline{\delta}(a_1+b_1), \overline{\delta}(a_2+b_2)\} &= (\overline{\delta} \times \overline{\delta})(a_1+b_1, a_2+b_2) \\ &= (\overline{\delta} \times \overline{\delta})((a_1,a_2)+(b_1,b_2)) \\ &\geq \min\{\overline{\delta} \times \overline{\delta})(a_1,a_2), (\overline{\delta} \times \overline{\delta})(b_1,b_2)\} \\ &= \min\{\min\{\overline{\delta}(a_1), \overline{\delta}(a_2)\}, \min\{\overline{\delta}(b_1), \overline{\delta}(b_2)\}\}. \end{aligned}$$

Now, putting $a_1 = x, a_2 = 0, b_1 = y$ and $b_2 = 0$, As $\overline{\delta}(0) \geq \overline{\delta}(x)$ for all $x \in S$, we see $\overline{\delta}(x+y) \geq \min\{\overline{\delta}(x), \overline{\delta}(y)\}$. Next, we have

$$\begin{aligned}\min\{\overline{\delta}(a_1b_1), \overline{\delta}(a_2b_2)\} &= (\overline{\delta} \times \overline{\delta})(a_1b_1, a_2b_2)\\ &= (\overline{\delta} \times \overline{\delta})((a_1, a_2)(b_1, b_2))\\ &\geq (\overline{\delta} \times \overline{\delta})(b_1, b_2)\\ &= \min\{\overline{\delta}(b_1), \overline{\delta}(b_2)\}.\end{aligned}$$

Taking $a_1 = x, b_1 = y$ and $b_2 = 0$, we obtain $\overline{\mu}(xy) \geq \overline{\mu}(y)$. Also if $(a_1, a_2) \leq (b_1, b_2)$, then $\min\{\overline{\mu}(a_1), \overline{\mu}(a_2) \geq \min\{\overline{\mu}(b_1), \overline{\mu}(b_2)\}$.

Now, putting $a_1 = x, a_2 = 0, b_1 = y$ and $b_2 = 0$, in this inequality we get $\overline{\mu}(x) \geq \overline{\mu}(y)$. Eventually $\overline{\mu}$ is an IVFLOI of $S$. □

The following result states that $\circ_{\overline{1}}$ is distributive over $+_{\overline{1}}$ addition.

**Proposition 3.5.** *For* $\overline{\delta}_1, \overline{\delta}_2, \overline{\delta}_3 \in \mathbb{A}$, $\overline{\delta}_1 \circ_{\overline{1}} (\overline{\delta}_2 +_{\overline{1}} \overline{\delta}_3) = (\overline{\delta}_1 \circ_1 \overline{\delta}_2) +_{\overline{1}} (\overline{\delta}_1 \circ_{\overline{1}} \overline{\delta}_3)$.

**Theorem 3.5.** *If* $\overline{\mu}_1, \overline{\mu}_2$ *are IVFOIs of* $S$, *then* $\overline{\mu}_1 +_{\overline{1}} \overline{\mu}_2$ *is so.*

*Proof.* The proof is clear. □

The following theorem shows $\circ_{\overline{1}}$ is closed in $\mathbb{D}$.

**Theorem 3.6.** *For* $\overline{\mu}_1, \overline{\mu}_2 \in \mathbb{D}$, $\overline{\mu} \circ_{\overline{1}} \overline{\mu}_2$ *is in* $\mathbb{D}$.

*Proof.* The Proof is trivial. □

**Theorem 3.7.** $\mathbb{D}$ *is zerosumfree semiring with infinite element 1 under the operations* $\circ_{\overline{1}}$ *and* $+_{\overline{1}}$.

*Proof.* Clearly, $\emptyset \in \mathbb{D}$. Let $\overline{\delta}_1, \overline{\delta}_2, \overline{\delta}_3 \in \mathbb{D}$. Then

(i) $\overline{\delta}_1 +_{\overline{1}} \overline{\delta}_2 \in IVFOI(S)$,
(ii) $\overline{\delta} \circ_{\overline{1}} \overline{\delta}_2 \in IVFOI(S)$,
(iii) $\overline{\delta} +_{\overline{1}} \overline{\delta}_2 = \overline{\mu}_2 +_1 \overline{\delta}_1$,
(iv) $\emptyset +_{\overline{1}} \overline{\delta}_1 = \overline{\delta}_1$,
(v) $\overline{\delta}_1 +_{\overline{1}} (\overline{\delta}_2 +_1 \overline{\mu}_3) = (\overline{\delta}_1 +_{\overline{1}} \overline{\delta}_2) +_1 \overline{\delta}_3$,
(vi) $\overline{\delta}_1 \circ_{\overline{1}} (\overline{\delta}_2 \circ_1 \overline{\delta}_3) = (\overline{\delta}_1 \circ_{\overline{1}} \overline{\delta}_2) \circ_{\overline{1}} \overline{\delta}_3$,
(vii) $\overline{\delta}_1 \circ_{\overline{1}} (\overline{\delta}_2 +_{\overline{1}} \overline{\delta}_3) = (\overline{\delta}_1 \circ_{\overline{1}} \overline{\delta}_2) +_{\overline{1}} (\overline{\delta}_1 \circ_{\overline{1}} \overline{\delta}_3)$,
(viii) $(\overline{\delta}_2 +_{\overline{1}} \overline{\delta}_3) \circ_1 \overline{1}\overline{\delta}_1 = (\overline{\delta}_2 \circ_{\overline{1}} \overline{\delta}_1) +_{\overline{1}} (\overline{\delta}_3 \circ_{\overline{1}} \overline{\delta}_1)$.

Also $\emptyset +_{\overline{1}} \overline{\delta}_1 = \overline{\delta}_1 +_{\overline{1}} L = \overline{\delta}_1$. Thus $\mathbb{D}$ is a semiring.

Now $1 \subseteq 1 +_{\overline{1}} \overline{\delta}_1$ for $\overline{\delta}_1 \in \mathbb{D}$.

Aslo $(1 +_{\overline{1}} \overline{\delta})(x) = \sup_{x \leq y+z}\{\min\{1(y), \overline{\delta}(z)\} | y, z \in S\} \leq 1(x)$ for all $x \in S$. Therefore $1 +_1 \overline{\delta}_1 \subseteq 1$ and hence $1 +_{\overline{1}} \overline{\delta}_1 = 1$ for all $\overline{\delta}_1 \in \mathbb{D}$. Thus 1 is an infinite element of $\mathbb{D}$.

Next let $\overline{\delta}_1 +_{\overline{1}} \overline{\delta}_2 = \emptyset$ for $\overline{\delta}_1, \overline{\delta}_2 \in \mathbb{D}$. Then $\overline{\delta}_1 \subseteq \overline{\delta}_1 +_{\overline{1}} \overline{\delta}_2 = \emptyset \subseteq \overline{\delta}_1$ and so $\overline{\delta}_1 = \emptyset$.

Similarly, it can be shown that $\overline{\delta}_2 = \emptyset$.

Hence $\mathbb{D}$ is zero sum free. □

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