Advances in Mathematics: Scientific Journal **9** (2020), no.4, 1921–1931
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.4.51 Spec. Issue on NCFCTA-2020

ON VON NEUMANN REGULAR MODULES

G. N. SUDHARSHANA[1] AND D. SIVA KUMAR

ABSTRACT. Results gotten for a module M over a commutative ring have been broadened to module over a ring which is not necessarily commutative. It has been indicated that an R-module M is VN-regular module if and only if M is a multiplication module and $R/(0:M)$ is strongly regular ring. It has also been indicated that the notions of prime submodule, completely prime submodule, maximal submodule coincide in a strong symmetric VN-regular module.

1. INTRODUCTION

In this manuscript, we develop the outcomes admitted for a VN-regular module over a commutative ring to a VN-regular module over a ring which is not necessarily commutative. Following [2], an element $a \in R$ is said to M-VN-regular if $aM = a^2M$ where R is a commutative ring and M is an R-module, respectively. Since R is commutative, $aM =< a >^2 M$ if and only if an element $a \in R$ is M-VN-regular, where $< a >$ is the ideal generated by a. An R-module M is said to be VN-regular module if for any $m \in M$, $Rm = aM$ for some $a \in R$, where a is a M-VN-regular element. We present the VN-regular modules over rings definitions which are not necessarily commutative and obtain the necessary and sufficient condition for an R-module M to be VN-regular module in Section 2.

[1]*corresponding author*

2010 *Mathematics Subject Classification.* 06F25.
Key words and phrases. Strongly regular ring, Module, Von Neumann regular element and module, strong symmetric module.

In this manuscript, all rings are with nonzero identity and all modules are nonzero unital. The ring R is said to be regular if given $a_1 \in R$, we can find a_2 in R in a ways that $a_1 = a_1a_2a_1$. The ring R is said to be strongly regular if given $a_1 \in R$, we can find a_2 in R in a ways that $a_1 = a_2a_1^2$. The two notions of regular and strongly regular coincide if R is a commutative ring. since R is regular and idempotents are central if and only if a ring R is strongly regular .

In recent years, some significant scientific results about several types of module had been accounted, see [3]-[7]. Anderson et al.[3] called VN-regular module as JT-regular module (Jayaraman and Ticker) and weakly JT-regular module if every $a \in R$ is M-VN-regular. In between these modules, they have shown that there are two other regular modules, namely strongly F-regular and F-regular. In fact they have shown that a module M is JT-regular which implies that M is strongly F-regular. It follows that M is F-regular which implies that M is weakly JT regular.

In this manuscript, we follow the notation given in [2] and develop the outcomes gotten by [2] for modules over commutative rings to modules over rings which are not necessarily commutative. We have given illustrations of VN-regular modules over ring which is not commutative. Throughout this manuscript R stands for a ring which is not necessarily commutative unless otherwise specified and M stands for an R-module. The ideal $(A : B)$ is represented by $(A : B) = \{a \in R : aB \subseteq A\}$, where A and B are any two submodules of M. The annihilator of M is denoted by $(0 : M)$. A of M is called proper if $A \neq M$. A definition of a maximal submodule is that a proper submodule A of M is not consists in any other proper submodule of M. P is completely prime if $a \in R$,$m \in M$, such that $am \in P$, where P is proper submodule, then we have $m \in P$ or $aM \subseteq P$. P of M is said to be a prime submodule if for all ideals I of R and submodules A of M such that $IA \subseteq P$, we have $A \subseteq P$ or $IM \subseteq P$. If M is a module over R, where R signifies a commutative ring then the two notions, completely prime submodule and prime submodule coincide.

Every submodule of M is of the form IM then a module M is called a multiplication module, for some ideal I of R. If there exists a submodule B of M such that $A + B = M$ and $A \cap B = 0$, then a submodule A of M is called a complemented submodule. $\mathcal{L}(R)$ and $\mathcal{L}(M)$ signifies the lattice of all ideals of R and the lattice of all submodules of M, respectively.

2. Characterizations of VN-regular modules

Definition 2.1. *An element $a \in R$ is said to be M-VN-regular if $aM =< a >^2 M$, where Let M is an R-module.*

Definition 2.2. *If for any $m \in M$, $Rm = aM$ for some $a \in R$ then b R-module M is called VN-regular module, where a is a M-VN-regular element.*

Now we provide a counter examples VN-regular module over a ring which is not commutative.

Example 1. *Let*

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \Big/ a, b, c \in Z_2 \right\}$$

be the ring with usual matrix addition and matrix multiplication. Then the R-module

$$M = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

is a VN-regular module as for

$$m = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$Rm = aM =< a >^2 M$ *where*

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

For other element in M the choice of a is obvious.

Example 2. *Consider the ring R as in the example 2.3. Then the R-module R_R is not a VN-regular module as for*

$$m = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

$Rm \neq aM =< a >^2 M$ *where*

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

There does not exist a in R such that$Rm = aM =< a >^2 M$.

Definition 2.3. *If $f - f^2 \in (0 : M)$, then $f \in R$ is called weak idempotent element.*

Lemma 2.1. *Let M be an R-module. If $R/(0 : M)$ is strongly regular, then for any $r \in R$, $a \in R$ and for all $m \in M$, there exist $r^{'} \in R$ such that $ram = ar^{'}m$.*

Proof. Let $a \in R$. Suppose $R/(0 : M)$ is strongly regular. Then there exist $\bar{b} \in R/(0 : M)$ such that $\bar{a} = \bar{b}\bar{a}^2$. It follows that $\bar{a} = \bar{a}\bar{b}\bar{a}$. Let $r \in R$. Since $\bar{a}\bar{b}$ is central we have $\bar{r}\bar{a} = \bar{r}(\bar{a}\bar{b}\bar{a}) = (\bar{a}\bar{b})\bar{r}\bar{a} = \bar{a}\bar{r^{'}}$ for some $\bar{r^{'}} = \bar{b}\bar{r}\bar{a} \in R/(0 : M)$. Then $ram = ar^{'}m$ for all $m \in M$. □

Lemma 2.2. *Let $R/(0 : M)$ be strongly regular and M be an R-module. If for any element a in R, we have $aM =< a > M$.*

Proof. Suppose $R/(0 : M)$ is strongly regular. Let $a \in R$. It is obvious that $aM \subseteq< a > M$. Let $x \in< a > M$. Then x can be written as $x = \sum_{i,j} r_i a r_j m_i$, where the sum is finite, for some $r_i, r_j \in R$ and $m_i \in M$. Then $x = \sum_i r_i a m_i^{'}$ for some $m_i^{'} = r_j m \in M$. Thus $x = \sum_i a r_i^{'} m_i^{'}$ by Lemma 2.1. Hence $< a > M \subseteq aM$ and $aM =< a > M$ holds. □

Lemma 2.3. *Let $R/(0 : M)$ be strongly regular and let $f_1, f_2 \in R$ be weak idempotent elements of R, Then*

(i) $1 - f_1,\ f_1 f_2,\ f_1 + f_2(1 - f_1)$ *are weak idempotent elements of R.*
(ii) $f_1 M \cap aM = f_1 aM\ \forall\ a \in R$.
(iii) $f_1 M + f_2 M = (f_1 + f_1(1 - f_1))M$.
(iv) $f_1 M = f_2 M \Longleftrightarrow (f_1) + (0 : M) = (f_2) + (0 : M)$
(v) $f_1 M$ *has a complement in $L(M)$.*

Proof.

(i) Let $f_1, f_2 \in R$ be any weak idempotent elements of R. Then $\bar{f_1}, \bar{f_1}$ are idempotent elements of $R/(0 : M)$, so $\overline{1 - f_1}$ is idempotent element of $R/(0 : M)$. Since $\bar{f_1}$ is central $(\bar{f_1}\bar{f_2})^2 = \bar{f_1}(\bar{f_2}\bar{f_1})\bar{f_2} = \bar{f_1}^2\bar{f_2}^2 = \bar{f_1}\bar{f_2}$. Hence $\bar{f_1}\bar{f_2}$ is an idempotent element of $R/(0 : M)$. As $\bar{f_1}$ is central, we have $(\overline{f_1 + f_2(1 - f_1)})^2 = \overline{f_1 + f_2(1 - f_1)}$. It follows that $1 - f_1, f_1 f_2, f_1 + f_2(1 - f_1)$ are weak idempotent elements of R.

(ii) Let $f_1 am \in f_1 aM$. Then $f_1 am = af_1^{'}m \in aM$, by Lemma 2.1. Hence $f_1 aM \subseteq f_1 M \cap aM$. Since $f_1 - f_1^2 \in (0 : M)$, we get $f_1 m = f_1^2 m$ for all $m \in M$. Let $m_1 \in f_1 M \cap aM$. This implies $m_1 = f_1 m^{'}$ and $m_1 = am^{"}$ for some $m^{'}$, $m^{"} \in M$. Thus $m_1 = f_1 m^{'} = f_1^2 m^{'} = f_1(f_1 m^{'}) = f_1 m_1$ and since $f_1 m_1 = f_1 am^{"}$, we have $m_1 = f_1 am^{"}$. Thus $m_1 \in f_1 aM$. Therefore $f_1 aM = f_1 M \cap aM$.

(iii) Obviously $(f_1 + f_1(1 - f_1))M \subseteq f_1M + f_2M$. Let $f_1m \in f_2M$. It is clear that $\bar{f_1} = \bar{f_2}^2 + \bar{f_2}(\bar{1} - \bar{f_1})\bar{f_1} = \bar{f_1}^2 + \bar{f_1}\bar{f_2}(\bar{1} - \bar{f_1})$ since $\bar{f_1}$ is central. It follows that $f_1M \subseteq f_1(f_1 + f_2(1 - f_1))M$. Hence $f_1M = f_1(f_1 + f_2(1 - f_1))M$. Hence $f_1M = f_1(f_1 + 2(1 - f_1))M = (f_1 + f_2(1 - f_1))f_1M \subseteq (f_1 + f_2(1 - f_1))M$. Similarly $f_2M \subseteq (f_1 + f_2(1 - f_1))M$. It follows that $f_1M + f_2M \subseteq (f_1 + f_2(1 - f_1))M$. Hence $f_1M + f_2M = (f_1 + f_2(1 - f_1))M$ holds.

(iv) Assume that $f_1M = f_2M$. As in Lemma 1(iv)[2], $< f_1 > +(0 : M) =< f_2 > +(0 : M)$ holds.

Conversely, now to claim $f_1M = f_2M$. Let $f_1m \in f_1M$ since $f_1 \in< f_1 > +(0 : M)$, it follows from the assumption that $f_1 = \sum_{i,j} r_i f_2 r_j + x$ for some $r_i, r_j \in R$ and $x \in (0 : M)$. Then $f_1m = \sum_{i,j} r_i f_2 r_j m = \sum_i r_i f_2 m'$ for some $m' = r_j m \in M$. Hence $f_1m = \sum_i f_2 r_i' m'$ by Lemma 2.1. It follows that $f_2M \subseteq f_2M$ and similarly $f_2M \subseteq f_1M$. Hence $f_1M = f_2M$ holds.

(v) Let $m \in M$. Then $m = 1.m = (f_1 + (1 - f_1))m \in f_1M + (1 - f_1)M$. Hence $f_1M + (1 - f_1)M = M$. Since by(ii) $f_1M \cap (1 - f_1)M = f_1(1 - f_1)M = 0$. Hence f_1M has a complement in $L(M)$. □

Lemma 2.4. *Suppose $R/(0 : M)$ is strongly regular then for every $a \in R$ we have $aM = eM$ for some weak idempotent element e in R.*

Proof. Let $a \in R$. For $\bar{a} \in R/(0 : M)$ there exists $\bar{b} \in R/(0 : M)$ such that $\bar{a} = \bar{b}\bar{a}^2$. Hence $\bar{a} = \bar{a}\bar{b}\bar{a}$ and it follows that ab is a weak idempotent in R. Clearly $abM \subseteq aM$. Let $am \in aM$. Since $\bar{a} = \bar{a}\bar{b}\bar{a}$, it follows that $(a - aba)m = 0$ for all $m \in M$. Hence $am = abam \in abM$. Therefore $aM = abM$ for some weak idempotent ab in R. □

Lemma 2.5. *Suppose J_1, J_2 be any two ideals of R such that $J_1 + J_2 = R$ and $J_1J_2 \subseteq (0 : M)$, where M is an R-module. Then the subsequent axioms are satisfies:*

(i) $J_1 + (0 : M) =< f_1 > +(0 : M)$ for some $f_1 \in J_1$

(ii) $J_2 + (0 : M) =< 1 - f_1 > +(0 : M)$ for some $(1 - f_1) \in J_2$

(iii) $J_1M =< f_1 > M$ and $J_2M =< 1 - f_1 > M$ for some f_1 and $(1 - f_1)$ such that $f_1 \in J_1$ and $(1 - f_1) \in J_2$.

Proof.

(i) Let $J_1 + J_2 = R$, there exist $i \in J_1$ and $j \in J_2$ such that $i + j = 1$. As $i(1 - i) = (1 - j)j = ij \in (0 : M)$ this implies that i,$(1 - i)$ and j,$(1 - j)$ are weak idempotent elements of R. It is clear that $< i > \subseteq J_1$. Let $r \in J_1$. Then $r = r(i+j) = ri + rj \in < i > + J_1 J_2 \subseteq < i > + (0:M)$. Hence $< i > + (0:M) = J_1 + (0:M)$ for some weak idempotent $i \in J_1$.

The proof of (ii) is same as proof of (i).

(iii) Let $\sum_k i_k m_k \in J_1 M$, where the sum is finite. As $i_k \in J_1 + (0:M)$, by (i) it follows that $i_k = \sum_{i,j} r_i f_1 r_j + x$ for some r_i,$r_j \in R$ and $x \in (0:M)$. Then $i_k m_k = \sum_{i,j} r_i f_1 r_j m_k \in < f_1 > M$. Thus $J_1 M \subseteq < f_1 > M$. Let $y \in < f_1 > M$. As $f_1 \in < f_1 > + (0:M)$ by (i) $f_1 = i + x^{'}$ for some $i \in I$, $x^{'} \in (0:M)$. Then $y = \sum_{i,j} r_i f_1 r_j m = \sum_{i,j} r_i (i + x^{'}) r_j m \in < i > M \subseteq J_1 M$. It follows that $J_1 M = < f_1 > M$ for some $f_1 \in J_1$. Similarly $J_2 M = < 1 - f_1 > M$ for some weak idempotent $(1 - f_1) \in J_2$. □

Definition 2.4. *An R-module M is said to be strong symmetric if for any a, $b \in R$, $m \in M$ such that $abm = bam$.*

Note: If R is a commutative ring, every R-module M is strong symmetric. There exist R-module M which is strong symmetric even though R is not commutative ring.

Now we give an illustration of a strong symmetric module.

Example 3. *The Module in Example 1 is strong symmetric module even though R is not commutative ring.*

Lemma 2.6. *Assume M is a strong symmetric R-module and let $f_1, f_2 \in R$ be any two weak idempotent elements of R. Then $f_1 M + f_2 M = (f_1 + f_2(1 - f_1))M$.*

Proof. Obviously, $(f_1 + f_2(1 - f_1))M \subseteq f_1 M + f_2 M$. Let $f_1 m \in f_1 M$. It is clear that $\bar{f_1} = \bar{f_1}^2 + \bar{f_2}(\overline{1 - f_1})\bar{f_1}$ as $\bar{v} \in R/(0:M)$. It follows that $f_1 m = (f_1^2 + f_1(1 - f_1))m$ for all $m \in M$ as M is strong symmetric.

This shows that $f_1 M \subseteq f_1(f_1 + f_2(1 - f_1))M$ and hence
$f_1 M = f_1(f_1 + f_2(1 - f_1))M = (f_1 + f_2(1 - f_1))f_1 M$ as M is strong symmetric. Hence $f_1 M = (f_1 + f_2(1 - f_1))f_1 M \subseteq (f_1 + f_2(1 - f_1))M$. Similarly $f_2 M \subseteq (f_1 + f_2(1 - f_1))M$ and therefore $f_1 M + f_2 M \subseteq (f_1 + f_2(1 - f_1))M$.

This shows that $f_1 M + f_2 M = (f_1 + f_2(1 - f_1))M$. □

The subsequent theorem finds the condition under which any element $a \in R$ to be M-VN-regular element.

Theorem 2.1. *$a \in R$ is M-VN-regular if $R/(0:M)$ is strongly regular.*

Proof. Since $R/(0:M)$ is strongly regular, we have for $\bar{a} \in R/(0:M)$ there exists $\bar{b} \in R/(0:M)$ such that $\bar{a} = \bar{a}^2\bar{b}$. Then $a - a^2b \in (0:M)$. This implies $a - a^2b = x$ for some $x \in (0:M)$, we have $a = a^2b + x \in < a^2 > +(0:M)$. Hence $< a > +(0:M) \subseteq < a^2 > +(0:M)$. Therefore, we have $< a > +(0:M) = < a^2 > +(0:M)$.

Let $am \in aM$. Since $a \in < a > +(0:M)$, we have $a = \sum_{i,j} r_i a^2 r_j + x'$ for some r_i,$r_j \in R$ and $x' \in (0:M)$. Then $am = \sum_{i,j} r_i a^2 r_j m = \sum_i r_i a^2 m_i'$ for some $m_i' = r_j m \in M$. We have $am = \sum_i a^2 r_i' m_i'$ by Lemma 2.1. This implies $aM \subseteq < a >< a > M$. Clearly $< a >< a > M \subseteq < a > M$. By Lemma 2.2 $< a >^2 M \subseteq aM$. Hence $aM = < a >^2 M$. □

Theorem 2.2. *Suppose $R/(0:M)$ is strongly regular. Then the following conditions are equivalent.*

(i) Every element of R is M-VN-regular.
(ii) $(J_1 \cap J_2)M = J_1J_2M \ \forall\ J_1, J_2 \in \mathcal{L}(R)$.
(iii) $J_1M = J_1^2M \ \forall\ J_1 \in \mathcal{L}(R)$.

Proof.
$(i) \implies (ii)$. Under condition (i) satisfied. Let $J_1, J_2 \in \mathcal{L}(R)$. Let $a \in R$. By (i) we have $aM = < a >^2 M$. Clearly $J_1J_2M \subseteq (J_1 \cap J_2)M$. Let $x \in (J_1 \cap J_2)M$. Then $x = \sum_i a_i m_i$ where the sum is finite and for some $a_i \in J_1 \cap J_2$ and $m_i \in M$. Since $aM = < a >< a > M$, For any i, $a_i m_i = \sum_n (\sum_{i,j} r_i a r_j)(\sum_{k,l} r_k a r_l) m_n$ for some r_i,r_j,r_k,$r_l \in R$ and $m_n \in M$. Hence by Lemma 2.1, $a_i m_i = a^2 m_p$ for some $m_p \in M$. Thus $x = a.am_p' \in J_1J_2M$ since $a \in J_1 \cap J_2$. This implies that $(J_1 \cap J_2)M \subseteq J_1J_2M$ and hence $(J_1 \cap J_2)M = J_1J_2M$ holds.

$(ii) \implies (iii)$. Under condition (ii) satisfied. Let $J_1 \in \mathcal{L}(R)$. It follows by (ii) that $J_1M = (J_1 \cap J_1)M = J_1^2M$.

$(iii) \implies (i)$. Under condition (iii) satisfied. Let $a \in R$, then by (iii), $< a > M = < a >^2 M$. Hence by Lemma 2.2, we have $aM = < a >^2 M$. □

Lemma 2.7. *[1] Let M be a finitely generated strong symmetric R-module and let I be an ideal of R such that $IM = M$ then there exists $x \equiv 1(modI)$ such that $xM = 0$.*

Proof. Suppose M has two generators. Let m_1,m_2 be the generators of M. Since $m_1 \in IM$, $m_1 = i_1 m'$ where $i_1 \in I$, $m' \in M$. As $m' \in M$, $m' = \beta_{11}m_1 + \beta_{12}m_2$ for some β_{11},$\beta_{12} \in R$. So $m_1 = i_1(\beta_{11}m_1+\beta_{12}m_2) = (i_1\beta_{11})m_1+(i_1\beta_{12})m_2$. Therefore

$$m_1 = i_{11}m_1 + i_{12}m_2 \tag{2.1}$$

for some $i_{11} = i_1\beta_{11} \in I$, $i_{12} = i_1\beta_{12} \in I$. Again, Since $m_2 \in IM$, $m_2 = i_2m''$ where $i_2 \in I$, $m'' \in M$. As $m'' \in M$, $m'' = \beta_{21}m_1 + \beta_{22}m_2$ for some β_{21},$\beta_{22} \in R$. So $m_2 = i_2(\beta_{21}m_1 + \beta_{22}m_2) = (i_2\beta_{21})m_1 + (i_2\beta_{22})m_2$. Therefore

$$m_2 = i_{21}m_1 + i_{22}m_2 \tag{2.2}$$

for some $i_{21} = i_2\beta_{21} \in I$, $i_{22} = i_2\beta_{22} \in I$. From (2.1),

$$(1 - i_{11})m_1 - i_{12}m_2 = 0. \tag{2.3}$$

From (2.2),

$$-i_{21}m_1 + (1 - i_{22})m_2 = 0\,. \tag{2.4}$$

Let $x = (1 - i_{11})(1 - i_{22}) - i_{12}i_{21}$. Then $xm_1 = ((1 - i_{11})(1 - i_{22}) - i_{12}i_{21})m_1 = (1 - i_{22})(1 - i_{11})m_1 - i_{12}i_{21}m_1$ since M is strong symmetric. By (2.3), $xm_1 = (1 - i_{22})(i_{12}m_2) - i_{12}i_{21}m_1 = (1 - i_{22})(i_{12}m_2) - i_{12}((1 - i_{22})m_2)$ by (2.4). Since M is strong symmetric, we have $xm_1 = 0$. Similarly $xm_2 = 0$. Let $m \in M$, $m = \alpha_1m_1 + \alpha_2m_2$ for some α_1, $\alpha_2 \in R$. Then $xm = x(\alpha_1m_1 + \alpha_2m_2) = 0$ since M is strong symmetric and $xm_1 = xm_2 = 0$. Hence $xm = 0$ for all $m \in M$.Thus $xM = 0$. We write $(1 - y)M = 0$ where $y \in I$ since x is of the form $x = (1 - i_{11})(1 - i_{22}) - i_{12}i_{21}$. Hence for n generators, we can easily find

$$x = \begin{bmatrix} 1 - i_{11} & -i_{12} & . \;. \;. & -i_{1n} \\ -i_{21} & 1 - i_{22} & . \;. \;. & -i_{2n} \\ . & . & . \;. \;. & . \\ . & . & . \;. \;. & . \\ . & . & . \;. \;. & . \\ -i_{n1} & -i_{n2} & . \;. \;. & 1 - i_{nn} \end{bmatrix}$$

□

Now we find the necessary condition for an element $a \in R$ to be M-VN-regular.

Theorem 2.3. *Let M is a strong symmetric R-module and it is finitely generated. Then a is M-VN-regular if and only if $R/(0 : M)$ is strongly regular.*

Proof. Suppose $R/(0:M)$ is strongly regular. Let $a \in R$. Then by Theorem 2.1, we have a is M-VN-regular.

Conversely, suppose that a is M-VN-regular. Then $aM =< a >^2 M$. As M is strong symmetric, we have $< a > M =< a >^2 M$. By Lemma 2.7, $(1-r) < a > M = 0$ for some $r \in < a >$. It follows that $(1-r)am = 0$ for all $m \in M$. Then $(1-\sum_{i,j} r_i a r_j)am = 0$ for some r_i,$r_j \in R$. Since M is strong symmetric, we have $0 = a(1-\sum_{i,j} r_i a r_j)m = a(m-\sum_{i,j} r_i a r_j m) = a(m-\sum_{i,j} a r_j r_i m) = (a-a^2r')m$ for all $m \in M$ and for some $r' = \sum_{i,j} r_j r_i \in R$. It follows that $\bar{a} = \bar{a}^2\bar{r'}$ and hence $R/(0:M)$ is strongly regular. □

Lemma 2.8. *Let M is a strong symmetric R-module and it is finitely generated. Then $a \in R$ is M-VN-regular if and only if $aM =< e > M$ for some $e \in R$.*

Proof. Let a be M-VN-regular. According to Theorem 2.3, $R/(0:M)$ is strongly regular. Since by Lemma 2.4, we have $aM = eM$ for some $e \in R$. By Lemma 2.2, we have $aM =< e > M$ for some $e \in R$.

Conversely, suppose that $aM =< e > M$ for $e \in R$. As M is strong symmetric, one obtain $< a >^2 M =< a >< e > M =< a > eM = e < a > M = e^2M = eM = aM$. Therefore $aM =< a >^2 M$. □

Lemma 2.9. *Let M is a strong symmetric VN-regular R-module and it is finitely generated. Then $R/(0:M)$ is strongly regular.*

Proof. Let $b \in R$. Since M is finitely generated, we have $< b > M$ is also finitely generated. As M is strong symmetric, we have bM is finitely generated. Then $bM = \sum_{i=1}^{n} Rm_i$ for some m_1,m_2,...,$m_n \in M$. As M is a VN-regular module, for each i, there exists a M-VN-regular element $b_i \in R$ such that $Rm_i = b_iM$. According to Lemma 2.8, for each i, there exists $e_i \in R$ such that $b_iM =< e_i > M = e_iM$.

Now by utilizing Lemma 2.6, $\sum_{i=1}^{n} Rm_i = \sum_{i=1}^{n} b_iM = \sum_{i=1}^{n} e_iM = eM$ for some $e \in R$. So $bM = eM$. This implies $bM =< e > M$ for $e \in R$. By Lemma 2.8, we have $b \in R$ is M-VN-regular and hence by Theorem 2.3, we have $R/(0:M)$ is strongly regular. □

Lemma 2.10. *Suppose M is a multiplication R-module and $R/(0:M)$ is strongly regular. Then M is a VN-regular module.*

Proof. As Rm is finitely generated, this implies that $Rm = IM$ for some finitely generated ideal $I \subseteq (Rm:M)$. As $R/(0:M)$ is strongly regular, by Lemma

2.4, we have for any $a \in R$, $aM = eM$ for some weak idempotent element e in R and since I is finitely generated, $IM = \sum_{i=1}^{n} a_i M = \sum_{i=1}^{n} e_i M = fM$, since by Lemma 2.3(iii), for some weak idempotent element $f \in R$. Consequently, $Rm = (Rm : M)M = IM = fM$ for some weak idempotent element $f \in R$, and hence M is a VN-regular module. □

Theorem 2.4. *Let M is a strong symmetric R-module and it is finitely generated. Then the following conditions are equivalent.*

(i) M is a VN-regular module.

(ii) M is a multiplication module and $R/(0 : M)$ strongly regular.

Proof.

$(i) \Longrightarrow (ii)$ As M is a finitely generated strong symmetric VN-regular module, then by Lemma 2.9 it is clear that $R/(0 : M)$ is strongly regular.

We have for each $m \in M$, $Rm =< a >^2 M = aM$. Let A be a submodule of M. Let $x \in A$. Then $Rx = I_x M$ for some ideal I_x of R. Let $I = \sum_{x \in N} I_x$. Then $x \in I_x M \subseteq IM$, this implies that $A \subseteq IM$. Let $i \in I$ be such that $i = i_1 + i_2 + ... + i_n$(say). Then $im = i_1 m + i_2 m + ... + i_n m \in A$. This implies $IM \subseteq A$ and hence $A = IM$ implies that M is a multiplication module.

$(ii) \Longrightarrow (i)$ follows by Lemma 2.10. □

Lemma 2.11. *If M is a strong symmetric module then every prime submodule of M is a completely prime submodule of M.*

Proof. Let P be a prime submodule of M. Let $a \in R$, $m \in M$ such that $am \in P$. Since M is strong symmetric module, for any $r \in R$,$m \in M$ we have $arm = ram$. Let $< m >$ be a submodule generated by m. Then for any $x \in < m >$ we have $x = rm$ for some $r \in R$. Hence $ax = arm = ram \in P$. Hence $a < m > \subseteq P$. Since $a \in (P :< m >)$, an ideal, it follows that $< a >< m > \subseteq P$. As P is a prime submodule, we have $< m > \subseteq P$ or $< a > M \subseteq P$. Thus $m \in P$ or $aM \subseteq P$. Thus P is completely prime submodule. □

Lemma 2.12. *If M is a strong symmetric VN-regular module then every prime submodule of M is a maximal submodule of M.*

Proof. Let A be a prime submodule of M. Let B be a submodule such that $A \subset B$. Let $x \in B/A$. By definition 2.2, $Rx = aM =< a >^2 M$. For any $m \in M$, let $am \in aM$. Then $am \in a^2 M$ since M is a strong symmetric. Consequently $am = a^2 m'$ for some $m' \in M$. Thus $a(m - am') \in A$.

Since A is prime, $aM \subseteq A$ or $(m - am^{'}) \in A$. If $aM \subseteq A$ then $Rx \subseteq A$ implies that $x \in A$, a contradiction. So $(m - am^{'}) \in A$. Since $aM = Rx \subseteq B$, $am^{'} \in B$. Since $(m - am^{'}) \in A$, it follows that $(m - am^{'}) \in B$. As $am^{'} \in B$, we have $m \in B$. Thus $B = M$. Hence A is a maximal submodule of M. □

ACKNOWLEDGMENT

We thank Dr. P. Dheena, Professor, Department of Mathematics, Annamalai University, for his very valuable and inspiring guidance, continuous encouragement and support.

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DEPARTMENT OF MATHEMATICS
ANNAMALAI UNIVERSITY, CHIDAMBARAM-608002, TAMIL NADU, INDIA
E-mail address: sudharshanasss3@gmail.com

DEPARTMENT OF MATHEMATICS
ANNAMALAI UNIVERSITY, CHIDAMBARAM-608002, TAMIL NADU, INDIA