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A NEW APPROACH FOR SOLVING A SYSTEM OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATION

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ABSTRACT. In this paper, a new method for finding a solution of system of linear and non-linear fractional order partial differential equations is developed. By combining the Shehu Transform method and Iterative method we built a new way called "Iterative Shehu Transform method". This method gives solutions without any discretization and free from round-off errors, which reduces the numerical computations. Lastly, some examples are illustrating the effectiveness of this new method.

1. INTRODUCTION

Applications of Fractional Differential Equations (FDEs) spread over so many fields like Polymer Physics, diffusion theory, viscoelasticity,fluid mechanics [1–4] with the help of successful modeling of FDEs in last decades. So, it is necessary to find efficient technics for finding solution of FDEs. There are various methods in the literature includes Adomian Decomposition method (ADM) [5], Variational Iteration Method (VIM) [6, 7], Laplace Decomposition Method (LDM) [8, 9], Laplace-Carson Decomposition Method (LCDM) [11], Homotopy Analysis Method (HAM) [11], Homotopy Pertubation Method (HPM) [12]. One of the efficient ways to solve so many nonlinear fractional orders Ordinary and Partial Differential Equations is Iterative methods.

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This Paper contains a new method which we call Iterative Shehu transform method (ISTM). This method provides the solution in minimum steps of iterations which lead to the solutions which are efficient and reliable. This method is the combination of Shehu transform method and Iterative method for finding the solution for system of fractional order partial differential equations. Proposed method ISTM gives the solutions without Discretization or transformation or restrictive assumption and free from round-off errors. Also gives an analytical solution by using the initial conditions and the boundary conditions is used to justify the results.

Authors also work on the generalize integral transform and generalization of different integral transforms like Laguerre transform [13], Hankel type transform [14] with their multiplicity [15] in different dimensions.

In this paper ISTM is used for solving system of nonlinear fractional order partial differential equations with several examples to verify the efficiency and reliability of the method.

2. Basic Definitions

First, we summaries definitions and some results dealing with fractional calculus [2, 3] and Shehu Transform which is needed for to understand this paper.

Definition 2.1. *[16] A real function $\Omega(p), p > 0$ is said to be in the space $C_\alpha, \alpha \in R$ if there exist areal number $t(> \alpha)$, such that $\Omega(p) = p^t\Omega_1(p)$ where $\Omega_1 \in C[0, \infty]$. So it clear that C_α contained in C_β if $\beta \leq \alpha$.*

Definition 2.2. *[16] A function $\Omega(p), p > 0$ is said to be in the space $C^m_\alpha, m \in N \cup \{0\}$, if $\Omega^{(m)} \in C_\alpha$.*

Definition 2.3. *[2, 17] The left side Riemann-Liouville fractional integral of order $\eta \geq 0$, of a function $\Omega \in C_\alpha, \alpha \geq -1$ is given as,*

$$I^\eta\Omega(p) = \begin{cases} \frac{1}{\Gamma(\eta)} \int_0^p \frac{\Omega(s)}{(p-s)^{1-\eta}} ds, & \eta > 0, p > 0 \\ \Omega(p), & \eta = 0. \end{cases}$$

Definition 2.4. *[2, 17] The left side Caputo fractional derivative of a function $\Omega \in C_\alpha^m$ of order $\eta \geq 0$, $\alpha \geq -1$ and $m \in \mathbb{N} \cup \{0\}$ is given as,*

$$D^\eta \Omega(p) = \frac{\partial^\eta \Omega(p)}{\partial p^\eta} = \begin{cases} I^{m-\eta}\left[\frac{\partial^m \Omega(p)}{\partial p^m}\right], & m-1 < \eta < m, m \in \mathbb{N}, \\ \frac{\partial^m \Omega(p)}{\partial p^m} & \eta = m. \end{cases}$$

Definition 2.5. *[10, 17] Mittag-Leffler Function is the generalization of exponential function denoted by $E_\alpha(z)$ (for one parameter), $E_{\alpha,\beta}(z)$ (for two parameter) defined as,*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad \alpha \in \mathbb{R}^+, z \in \mathbb{C}$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad \alpha, \beta \in \mathbb{R}^+, z \in \mathbb{C}\,.$$

Definition 2.6. *[18] The Shehu transform of $\Omega(p)$ is defined as*

$$\mathcal{S}[\Omega(p)] = \Omega^*(x,y) = \int_0^\infty \exp(\frac{-xp}{y})\Omega(p)dp\,.$$

Eventually,

$$\Omega(p) = \mathcal{S}^{-1}\Omega^*(x,y) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{1}{y}\exp(\frac{xp}{y})\Omega^*(x,y)dx\,,$$

where x and y are Shehu Transform variables, with δ be a real constant.

Definition 2.7. *[19] The Shehu Transform $S[\Omega(p)]$, of the Riemann-Liouville fractional integral is given by,*

$$\mathcal{S}[I^\eta \Omega(p)] = \left(\frac{x}{y}\right)^{-\eta} \Omega^*(x,y).$$

Definition 2.8. *[19] The Shehu Transform $S[\Omega(p)]$, of the Caputo fractional derivative is given by,*

$$\mathcal{S}[D^\eta \Omega(p)] = \left(\frac{x}{y}\right)^{\eta} \Omega^*(x,y) - \sum_{k=0}^{n-1} \left(\frac{x}{y}\right)^{\eta-k-1} \Omega^{(k)}(0), \quad n-1 < \eta < n. \tag{2.1}$$

3. ITERATIVE SHEHU TRANSFORM METHOD

To express the idea of this method, consider the system of fractional order partial differential equations (FPDEs) with initial conditions of the form:

$$D_r^{\alpha_i}\mu_i[p,r] = A_i[\mu_1[p,r],\mu_2[p,r]...\mu_n[p,r]], \tag{3.1}$$

$m_i - 1 < \alpha_i \leq m_i$, $i = 1,2,...n$,

$$\frac{\partial^{(k_i)}\mu_i[p,r]}{\partial r^{k_i}} = \partial_{ik_i}(p), \tag{3.2}$$

$k_i = 0,1,2...m_{i-1}$, $m_i \in N$, where A_i is the nonlinear operators and $\mu_i[p,r]$ are the unknown functions. Taking the Shehu transform on both sides of equation (3.1) we get,

$$\mathcal{S}[D_r^{\alpha_i}\mu_i[p,r]] = \mathcal{S}[A_i[\mu_1[p,r],\mu_2[p,r]...\mu_n[p,r]]], \quad i = 1,2,...n. \tag{3.3}$$

From Definition 2.1 and the initial conditions (3.2) we have,

$$\left(\frac{x}{y}\right)^{\alpha_i}\mathcal{S}[\mu_i[p,r]] - \sum_{k=0}^{m_i-1}\left(\frac{x}{y}\right)^{\alpha_i-k-1}\mu_i^{(k)}(p,0) = \mathcal{S}[A_i[\mu_1[p,r],\mu_2[p,r]...\mu_n[p,r]]],$$

where $i = 1,2,...n$. Applying the definition of inverse Shehu transform on both sides of equation (3.3) we get,

$$\begin{aligned}\mu_i[p,r] &= \mathcal{S}^{-1}\Big[\sum_{k=0}^{m_i-1}\left(\frac{x}{y}\right)^{-k-1}\mu_i^{(k)}(p,0)\Big] + \mathcal{S}^{-1}\{\left(\frac{x}{y}\right)^{-\alpha_i}\mathcal{S}[A_i[\mu_1[p,r],...\mu_n[p,r]]]\} \\ &= \mu_i + N_i\{\mu_1[p,r],\mu_2[p,r]...\mu_n[p,r]\}, \quad i = 1,2,...n,\end{aligned}$$

which is written in the form

$$\mu_i[p,r] = \mu_i + N_i\{\mu_1[p,r],\mu_2[p,r]...\mu_n[p,r]\}, i = 1,2,...n,$$

where,

$$\mu_i = \mathcal{S}^{-1}\Big[\sum_{k=0}^{m_i-1}\left(\frac{x}{y}\right)^{-k-1}\mu_i^{(k)}(p,0)\Big] i = 1,2,...n,$$

$$N_i\{\mu_1[p,r],\mu_2[p,r]...\mu_n[p,r]\} = \mathcal{S}^{-1}\{\left(\frac{x}{y}\right)^{-\alpha_i}\mathcal{S}[A_i[\mu_1[p,r],...\mu_n[p,r]]]\}.$$

Now we find a solution u of equation (3.2) which is in series form

$$\mu_i[p,r] = \sum_{j=0}^{\infty}\mu_{ij}(p,r), i = 1,2,...n. \tag{3.4}$$

The nonlinear operator N_i can be written as,

$$N_i(\sum_{j=0}^{\infty}\mu_{1j}(p,r),...\sum_{j=0}^{\infty}\mu_{nj}(p,r)) = N_i(\mu_{10}(p,r),...\mu_{n0}(p,r)) + \sum_{j=1}^{\infty}\{N_i(\sum_{k=0}^{j}\mu_{1k}(p,r),...\sum_{k=0}^{j}\mu_{nk}(p,r)) - N_i(\sum_{k=0}^{j-1}\mu_{1k}(p,r),...\sum_{k=0}^{j-1}\mu_{nk}(p,r))\}. \tag{3.5}$$

From equation (3.4) and (3.5) equivalently written as,

$$\sum_{j=0}^{\infty}\mu_{ij}(p) = \mu_i + N_i(\mu_{10}(p,r),...\mu_{n0}(p,r)) + \sum_{j=1}^{\infty}\{N_i(\sum_{k=0}^{j}\mu_{1k}(p,r),...\sum_{k=0}^{j}\mu_{nk}(p,r)) - N_i(\sum_{k=0}^{j-1}\mu_{1k}(p,r),...\sum_{k=0}^{j-1}\mu_{nk}(p,r)\}. \tag{3.6}$$

By defining recurrent relation,

$$\mu_{i0}(p,r) = \mathcal{S}^{-1}\Big[\sum_{k=0}^{m_i-1}\left(\frac{x}{y}\right)^{-k-1}\mu_i^{(k)}(p,0)\Big]$$

$$\mu_{i1}(p,r) = \mathcal{S}^{-1}\Big[\left(\frac{x}{y}\right)^{-\alpha_i}\mathcal{S}\Big[A_i[\mu_{10}[p,r],...\mu_{n0}[p,r]]\Big]\Big]$$

$$\mu_{i(m+1)}(p,r) = \mathcal{S}^{-1}[\left(\frac{x}{y}\right)^{-\alpha_i}\mathcal{S}\Big[A_i\Big[[\mu_{10}[p,r]+...+\mu_{1m}[p,r]]... [\mu_{n0}[p,r]+...+\mu_{nm}[p,r]]]]] - \mathcal{S}^{-1}[\left(\frac{x}{y}\right)^{-\alpha_i} \mathcal{S}[A_i[[\mu_{10}[p,r]+...+\mu_{1(m-1)}[p,r]]... [\mu_{n0}[p,r]+...+\mu_{n(m-1)}[p,r]]]]]\Big]$$

Then,

(3.7)

$$\mu_{i1}(p,r)+...+\mu_{i(m+1)}(p,r)=$$
$$\mathcal{S}^{-1}\Big[\left(\frac{x}{y}\right)^{-\alpha_i}\mathcal{S}\Big[A_i\Big[[\mu_{10}[p,r],...\mu_{1m}[p,r]]... \quad [\mu_{n0}[p,r],...\mu_{nm}[p,r]]\Big]\Big]\Big].$$

So, the solution of equation (3.1) and (3.2) is obtained by approximating n-term given by,

$$\mu_i[p,r]\cong\mu_{i1}[p,r]+...+\mu_{in}[p,r], \quad i=1,2,...n.$$

This series solutions converges rapidly to exact solutions.

4. EXAMPLES

In this section, we test the applicability of iterative Shehu transform method for solving the systems of linear and nonlinear FPDEs.

Example 1. *Consider the system of linear FPDEs [11]:*

(4.1) $$D_r^{\alpha}\mu-\nu_p+\nu+\mu=0, \quad D_r^{\beta}\nu-\mu_p+\mu+\nu=0, \quad (0<\alpha,\beta\leq 1),$$

with initial conditions,

$$\mu(p,0)=\sinh(p), \quad \nu(p,0)=\cosh(p).$$

Exact solution, when $\alpha=\beta=1$, is

$$\mu(p,r)=\sinh(p-r), \quad \nu(p,r)=\cosh(p-r).$$

The system of linear FPDE's (4.1) *corresponding to the following Shehu equations:*

$$\mu(p,r)=\mathcal{S}^{-1}\Big[\left(\frac{x}{y}\right)^{-1}\mu(p,0)\Big]+\mathcal{S}^{-1}\Big[\left(\frac{x}{y}\right)^{-\alpha}\mathcal{S}\Big[\nu_p(p,r)-\nu(p,r)-\mu(p,r)\Big]\Big]$$
$$\nu(p,r)=\mathcal{S}^{-1}\Big[\left(\frac{x}{y}\right)^{-1}\nu(p,0)\Big]+\mathcal{S}^{-1}\Big[\left(\frac{x}{y}\right)^{-\alpha}\mathcal{S}\Big[\mu_p(p,r)-\nu(p,r)-\mu(p,r)\Big]\Big],$$

using the algorithm in equation (3.7) *for some terms of* $\mu(p,r)$ *and* $\nu(p,r)$ *are*

$$\mu_0(p,r) = \sinh(p), \quad \nu_0(p,r) = \cosh(p),$$
$$\mu_1(p,r) = -\frac{\cosh(p)r^{\alpha}}{\Gamma(\alpha+1)}, \quad \nu_1(p,r) = -\frac{\sinh(p)r^{\beta}}{\Gamma(\beta+1)},$$
$$\mu_2(p,r) = -\frac{\cosh(p)r^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\sinh(p)r^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\cosh(p)r^{2\alpha}}{\Gamma(2\alpha+1}$$
$$\nu_1(p,r) = -\frac{\sinh(p)r^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\cosh(p)r^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\sinh(p)r^{2\beta}}{\Gamma(2\beta+1)}.$$

The series solution is given by,

$$\begin{aligned}\mu(p,r) &= \mu_0(p,r) + \mu_1(p,r) + \mu_2(p,r) + ...\\ &= \sinh(p)\Big(1 + \frac{r^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + ...\Big)\\ &\quad - \cosh(p)\Big(\frac{r^{\alpha}}{\Gamma(\alpha+1)} + \frac{r^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{r^{2\alpha}}{\Gamma(2\alpha+1)} + ...\Big),\end{aligned} \tag{4.2}$$

$$\begin{aligned}\nu(p,r) &= \nu_0(p,r) + \nu_1(p,r) + \nu_2(p,r) + ...\\ &= \cosh(p)\Big(1 + \frac{r^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + ...\Big)\\ &\quad - \sinh(p)\Big(\frac{r^{\beta}}{\Gamma(\beta+1)} + \frac{r^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{r^{2\beta}}{\Gamma(2\beta+1)} + ...\Big),\end{aligned} \tag{4.3}$$

By putting $\alpha = \beta$ *in equation* (4.2) *and* (4.3), *we find the solution of equation* (4.1) *as follows:*

$$\begin{aligned}\mu(p,r) = \sinh(p)\Big(1 + \frac{r^{2\alpha}}{\Gamma(2\alpha+1)} + ...\Big) -\\ \cosh(p)\Big(\frac{r^{\alpha}}{\Gamma(\alpha+1)} + \frac{r^{3\alpha}}{\Gamma(3\alpha+1)} + ...\Big),\end{aligned} \tag{4.4}$$

$$\begin{aligned}\nu(p,r) = \cosh(p)\Big(1 + \frac{r^{2\alpha}}{\Gamma(2\alpha+1)} + ...\Big) -\\ \sinh(p)\Big(\frac{r^{\alpha}}{\Gamma(\alpha+1)} + \frac{r^{3\alpha}}{\Gamma(3\alpha+1)} + ...\Big),\end{aligned}$$

now by putting $\alpha = 1$ *in equation* (4.3) *and* (4.4)*, we get,*

$$\mu(p,r) = \sinh(p)\Big(1 + \frac{r^2}{\Gamma(3)} + \frac{r^4}{\Gamma(5)}...\Big) - \cosh(p)\Big(\frac{r}{\Gamma(2)} + \frac{r^3}{\Gamma(4)} + \frac{r^5}{\Gamma(6)}...\Big),$$

$$\nu(p,r) = \cosh(p)\Big(1 + \frac{r^2}{\Gamma(3)} + \frac{r^4}{\Gamma(5)}...\Big) - \sinh(p)\Big(\frac{r}{\Gamma(2)} + \frac{r^3}{\Gamma(4)} + \frac{r^5}{\Gamma(6)}...\Big),$$

$$i.e.\mu(p,r) = \sinh(p)\Big(1 + \frac{r^2}{2!} + \frac{r^4}{4!}...\Big) - \cosh(p)\Big(r + \frac{r^3}{3!} + \frac{r^5}{5!}...\Big) = \sinh(p-r),$$

$$\nu(p,r) = \cosh(p)\Big(1 + \frac{r^2}{2!} + \frac{r^4}{4!}...\Big) - \sinh(p)\Big(r + \frac{r^3}{3!} + \frac{r^5}{5!}...\Big) = \cosh(p-r),$$

which is the exact solution of (4.1) *by taking* $\alpha = \beta = 1$.

Example 2. *Consider the system of nonlinear FPDEs [11]:*

$$D_r^{\alpha}\mu + \nu_p\tau_q - \nu_q\tau_p = -\mu,$$
$$D_r^{\beta}\nu + \mu_p\tau_q + \mu_q\tau_p = \nu,$$
$$D_r^{\gamma}\tau + \mu_p\nu_q + \mu_q\nu_p = \tau, (0 < \alpha, \beta, \gamma \le 1), \tag{4.5}$$

with initial conditions,

$$\mu(p,q,0) = e^{p+q}, \quad \nu(p,q,0) = e^{p-q}, \quad \tau(p,q,0) = e^{-p+q}.$$

Exact solution, when $\alpha = \beta = \gamma = 1$*, is*

$$\mu(p,q,r) = e^{p+q-r}, \quad \nu(p,q,r) = e^{p-q+r}, \quad \tau(p,q,r) = e^{-p+q+r}.$$

Like Example 1, we write,

$$\mu(p,q,r) = \mathcal{S}^{-1}\Big[\Big(\frac{x}{y}\Big)^{-1}\mu(p,q,0)\Big] + \mathcal{S}^{-1}\Big[\Big(\frac{x}{y}\Big)^{-\alpha}$$
$$\mathcal{S}\Big[-\mu(p,q,r) - \nu_p(p,q,r)\tau_q(p,q,r) + \nu_q(p,q,r)\tau_p(p,q,r)\Big]\Big]$$

$$\nu(p,q,r) = \mathcal{S}^{-1}\Big[\Big(\frac{x}{y}\Big)^{-1}\nu(p,q,0)\Big] + \mathcal{S}^{-1}\Big[\Big(\frac{x}{y}\Big)^{-\beta}$$
$$\mathcal{S}\Big[\nu(p,q,r) - \mu_p(p,q,r)\tau_q(p,q,r) - \mu_q(p,q,r)\tau_p(p,q,r)\Big]\Big]$$

$$\tau(p,q,r) = \mathcal{S}^{-1}\Big[\Big(\frac{x}{y}\Big)^{-1}\tau(p,q,0)\Big] + \mathcal{S}^{-1}\Big[\Big(\frac{x}{y}\Big)^{-\gamma}$$
$$\mathcal{S}[\tau(p,q,r) - \mu_p(p,q,r)\nu_q(p,q,r) - \mu_q(p,q,r)\nu_p(p,q,r)\Big]\Big],$$

using the algorithm in equation (3.7) *for some terms of* $\mu(p,q,r), \nu(p,q,r)$ *and* $\tau(p,q,r)$ *are*

$$\mu_0(p,q,r) = e^{p+q}, \nu_0(p,q,r) = e^{p-q}, \tau_0(p,q,r) = e^{-p+q}$$

$$\mu_1(p,q,r) = -\frac{e^{p+q}r^{\alpha}}{\Gamma(\alpha+1)} - \frac{e^{p-q}e^{-p+q}r^{\alpha}}{\Gamma(\alpha+1)} + \frac{e^{p-q}e^{-p+q}r^{\alpha}}{\Gamma(\alpha+1)} = -\frac{e^{p+q}r^{\alpha}}{\Gamma(\alpha+1)},$$

$$\nu_1(p,q,r) = -\frac{e^{p-q}r^{\beta}}{\Gamma(\beta+1)} - \frac{e^{p+q}e^{-p+q}r^{\beta}}{\Gamma(\beta+1)} + \frac{e^{p+q}e^{-p+q}r^{\beta}}{\Gamma(\beta+1)} = -\frac{e^{p-q}r^{\beta}}{\Gamma(\beta+1)},$$

$$\tau_1(p,q,r) = -\frac{e^{-p+q}r^{\gamma}}{\Gamma(\gamma+1)} - \frac{e^{p+q}e^{p-q}r^{\gamma}}{\Gamma(\gamma+1)} + \frac{e^{p+q}e^{p-q}r^{\gamma}}{\Gamma(\gamma+1)} = -\frac{e^{-p+q}r^{\gamma}}{\Gamma(\gamma+1)},$$

$$\begin{aligned}
\mu_2(p,q,r) &= \frac{e^{p+q}r^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{e^{p-q}e^{-p+q}r^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} - \frac{e^{p-q}e^{-p+q}r^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\
&\quad - \frac{\Gamma(\gamma+\beta+1)e^{p-q}e^{-p+q}r^{\alpha+\beta+\gamma}}{\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1} + \frac{e^{p-q}e^{-p+q}r^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} \\
&\quad + \frac{e^{p-q}e^{-p+q}r^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\Gamma(\gamma+\beta+1)e^{p-q}e^{-p+q}r^{\alpha+\beta+\gamma}}{\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)}, \\
\mu_2(p,q,r) &= \frac{e^{p+q}r^{2\alpha}}{\Gamma(2\alpha+1)} \\
\nu_2(p,q,r) &= \frac{e^{p-q}r^{2\beta}}{\Gamma(2\beta+1)} - \frac{e^{p-q}e^{-p+q}r^{\gamma+\beta}}{\Gamma(\alpha+\gamma+1)} - \frac{e^{p+q}e^{-p+q}r^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\
&\quad - \frac{\Gamma(\gamma+\alpha+1)e^{p-q}e^{-p+q}r^{\alpha+\beta+\gamma}}{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1} + \frac{e^{p+q}e^{-p+q}r^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)} \\
&\quad + \frac{e^{p+q}e^{-p+q}r^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\Gamma(\gamma+\beta+1)e^{p+q}e^{-p+q}r^{\alpha+\beta+\gamma}}{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)}, \\
\nu_2(p,q,r) &= \frac{e^{p-q}r^{2\beta}}{\Gamma(2\beta+1)} \\
\tau_2(p,q,r) &= \frac{e^{-p+q}r^{2\gamma}}{\Gamma(2\gamma+1)} - \frac{e^{p+q}e^{p-q}r^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)} - \frac{e^{p+q}e^{p-q}r^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} \\
&\quad - \frac{\Gamma(\alpha+\beta+1)e^{p+q}e^{p-q}r^{\alpha+\beta+\gamma}}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+\gamma+1} + \frac{e^{p+q}e^{p-q}r^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)} \\
&\quad + \frac{e^{p+q}e^{p-q}r^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} + \frac{\Gamma(\alpha+\beta+1)e^{p+q}e^{p-q}r^{\alpha+\beta+\gamma}}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+\gamma+1)}, \\
\tau_2(p,q,r) &= \frac{e^{-p+q}r^{2\gamma}}{\Gamma(2\gamma+1)}
\end{aligned}$$

Therefore, the series solution is given by,

$$\begin{aligned}\mu(p,q,r) &= e^{p+q} - \frac{e^{p+q}r^{\alpha}}{\Gamma(\alpha+1)} + \frac{e^{p+q}r^{2\alpha}}{\Gamma(2\alpha+1)} + ... \\ &= e^{p+q}\Big(1 + \sum_{k=1}^{\infty}\frac{(-r^{\alpha})^k}{\Gamma(k\alpha+1)}\Big) = e^{p+q}E_{\alpha}(-r^{\alpha}), \\ \nu(p,q,r) &= e^{p-q} - \frac{e^{p-q}r^{\beta}}{\Gamma(\beta+1)} + \frac{e^{p-q}r^{2\beta}}{\Gamma(2\beta+1)} + ... \\ &= e^{p+q}\Big(1 + \sum_{k=1}^{\infty}\frac{(-r^{\beta})^k}{\Gamma(k\beta+1)}\Big) = e^{p-q}E_{\beta}(-r^{\beta}),\end{aligned}$$

$$\begin{aligned}\tau(p,q,r) &= e^{-p+q} - \frac{e^{-p+q}r^{\gamma}}{\Gamma(\gamma+1)} + \frac{e^{-p+q}r^{2\gamma}}{\Gamma(2\gamma+1)} + ... \\ &= e^{p+q}\Big(1 + \sum_{k=1}^{\infty}\frac{(-r^{\gamma})^k}{\Gamma(k\gamma+1)}\Big) = e^{-p+q}E_{\gamma}(-r^{\gamma}),\end{aligned}$$

By putting $\alpha = \beta = \gamma = 1$ we get,

$$\begin{aligned}\mu(p,q,r) &= e^{p+q} - \frac{e^{p+q}r}{\Gamma(2)} + \frac{e^{p+q}r^2}{\Gamma(3)} + ... \\ &= e^{p+q}\Big(1 - r + \frac{r^2}{2!} - \frac{r^3}{3!}...\Big) \\ &= e^{p+q-r}\end{aligned}$$

$$\begin{aligned}\nu(p,q,r) &= e^{p-q} + \frac{e^{p-q}r}{\Gamma(2)} + \frac{e^{p-q}r^2}{\Gamma(3)} + ... \\ &= e^{p-q}\Big(1 + r + \frac{r^2}{2!} + \frac{r^3}{3!}...\Big) \\ &= e^{p+q+r}\end{aligned}$$

$$\begin{aligned}T(p,q,r) &= e^{-p+q} + \frac{e^{-p+q}r}{\Gamma(2)} + \frac{e^{-p+q}r^2}{\Gamma(3)} + ... \\ &= e^{-p+q}\Big(1 + r + \frac{r^2}{2!} + \frac{r^3}{3!}...\Big) \\ &= e^{-p+q+r},\end{aligned}$$

which is an exact solution of (4.5) *by taking $\alpha = \beta = 1$. We also note that the obtained result is like the result obtained by VIM and HAM [6, 7, 11].*

5. CONCLUSION

In this paper, a new method called Iterative Shehu Transform method is developed and applied it for finding the exact and approximate solutions of fractional partial differential equations. This method gives solution in minimum computational work with high accuracy as compare to traditional classical methods. It is also seen that ISTM has some advantage as compare to Homotopy analysis method and Adomian decomposition methods because it does not requireany numerical computation when we solve nonlinear problems. From this we can clearly say that Iterative Shehu transformation method is the best refinement for existing methods. Finally, we have illustrated the applications of the method with help of solving two examples.

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