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ON SUBCLASSES OF UNIVALENT FUNCTIONS HAVING NEGATIVE COEFFICIENT USING RUSAL DIFFERENTIAL OPERATOR

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ABSTRACT. In this paper we introduced new classes $DA(\eta,\xi,\alpha,\beta,\partial,\lambda)$ and $DA(\eta,\xi,\alpha,\beta,\partial,\lambda,c)$ of normalized, analytic, univalent functions on unit disc Δ. Using Rusal differential operator and neighbourhood property we derived certain inclusions with given classes. Some examples on neighbourhood of identity functions are discussed. Two subclasses of $DA(\eta,\xi,\alpha,\beta,\partial,\lambda)$ are also identified.

1. INTRODUCTION

Let N be the class of normalized analytic, univalent functions on unit disc Δ with negative coefficient given by

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0). \tag{1.1}$$

Definition 1.1. *In [2] Al-Oboudi has introduced differential operator as follows* $D^n : N \to N$ *defined by*

$$D^0 f(z) = f(z).$$

$$D^1 f(z) = (1-\lambda) f(z) + z f^{'}(z) = D_\lambda f(z) \lambda \geq 0, \tag{1.2}$$

$$D^n f(z) = D_\lambda (D^{n-1} f(z)). \tag{1.3}$$

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From (1.2) *and* (1.3) *we have*

$$D^n(f(z)) = z + \sum_{k=2}^{\infty}[1+(k-1)\partial]^n a_k z^k \quad (z \in \Delta).$$

Definition 1.2. *For $f \in N$, in [3] following Ruschweyh differential operator is defined $R^n : N \to N$*

$$R^n(f(z)) = \frac{z}{(1-z)^{n+1}}.f(z) \quad n \in N \cup \{0\}$$
$$= z + \sum_{k=2}^{\infty} {}^{n+k-1}_{\ \ n}C a_k z^k \quad (z \in \Delta),$$

where $(.)$ *is Hadamard product defined in* (2.1).

We note that $R^0 f(z) = f(z), R' f(z) = zf'(z)$. [4] has used the following Rusal differential operator which is obtained by linear combination of Ruschweyh and Al-Oboudi differential operator.

Definition 1.3. *Let $n \in N \cup \{0\}, \lambda \geq 0, A^n_\lambda : N \to N$ defined by*

$$A^n_\lambda(f(Z)) = (1-\lambda)D^n f(Z) + \lambda R^n f(z).$$

After simplification we will get,

$$A^n_\lambda(f(z)) = z + \sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda) + \lambda\, {}^{n+k-1}_{\ \ n}C a_k z^k.$$

If $n = 0$, $A^0_\lambda f(z) = f(z)$.

[1] has used the class $AR(\eta, \xi, \alpha, \beta, \lambda)$. In the next section we discussed the class $DA(\eta, \xi, \alpha, \beta, \partial, \lambda)$ which is generalization of $AR(\eta, \xi, \alpha, \beta, \lambda)$.

2. Classes $DA(\eta, \xi, \alpha, \beta, \partial, \lambda)$ and $DA(\eta, \xi, \alpha, \beta, \partial, \lambda, \mathcal{C})$.

In this section we introduced with $DA(\eta, \xi, \alpha, \beta, \partial, \lambda)$ and $DA(\eta, \xi, \alpha, \beta, \partial, \lambda, \mathcal{C})$. We also defined the neighbourhood which will be useful in main results.

Definition 2.1. *For $f_1(z) = z - \sum_{k=2}^{\infty} a_k$ and $f_2(z) = z - \sum_{k=2}^{\infty} a_k$ in $DA(\eta, \xi, \alpha, \beta, \partial, \lambda)$ defined $*$ and $.$ as bellow*

$$h.f_1(z) = h.\left(z - \sum_{k=2}^{\infty} a_k z^k\right) = z - \sum_{k=2}^{\infty} h a_k z^k,$$

$$f_1(z) * f_2(z) = \left(z - \sum_{k=2}^{\infty} a_k z^k\right) * \left(z - \sum_{k=2}^{\infty} b_k z^k\right) = z - \sum_{k=2}^{\infty} (a_k b_k) z^k. \tag{2.1}$$

Definition 2.2. *A function $f(z)$ in N is said to be in $DA(\eta, \xi, \alpha, \beta, \partial, \lambda, \mathcal{C})$ if and only if*

$$\left| \frac{\frac{z(A_\lambda^n(f))'}{A_\lambda^n(f)} - 1}{2\xi\left(\frac{z(A_\lambda^n(f))'}{A_\lambda^n(f)} - \alpha\right) - \left(\frac{z(A_\lambda^n(f))'}{A_\lambda^n(f)} - 1\right)} \right| < \beta,$$

where $0 \leq \alpha < \frac{1}{2\xi}, 0 < \beta \leq 1, \frac{1}{2} \leq \xi \leq 1, n \in N \cup \{0\}$. Note that if $\lambda = 0$, then $DA(\eta, \xi, \alpha, , \beta, \partial, 0)$ is the class $AR(\eta, \xi, \alpha, \beta, \lambda)$ discussed by [1].

The remark given bellow is proven as a theorem in [5].

Remark 2.1. *If $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ is in N, then $f \in DA(\eta, \xi, \alpha, \beta, \partial, \lambda)$ if and only if*

$$\sum_{k=2}^{\infty} ([1+(k-1)\partial]^n (1-\lambda) + \lambda \, {}_n^{n+k-1}C)(2\xi\beta(k-\alpha) + (k-1)(1-\beta)) a_k < 2\xi\beta(1-\alpha).$$

Definition 2.3. *Let $\gamma \geq 0$ & $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in N$ then, (γ, P)-neighbourhood of the function $f(z)$ is denoted by $N_{\gamma,p} f$. It is defined as follows $N_{\gamma,p} f(z) = \{g \in N | \sum_{k=2}^{\infty} k|a_k - pb_k| \leq \gamma\}$. If $p = 1$, we get the neighbourhood defined by [1]. (k, γ, P)- neighbourhood of the function $f(z)$ is denoted by $N_{k,\gamma,p} f$ and it is defined as*

$$N_{k,\gamma,p} f(z) = \left\{ g(z) \in N | g(z) = z - \sum_{k=2}^{\infty} \frac{b_k}{k} z^k \right\}$$

and

$$\sum_{k=2}^{\infty} k \left| a_k - p \frac{b_k}{k} \right| \leq \gamma.$$

We note that $N_{k,\gamma,p} f(z) \subseteq N_{\gamma,p} f(z)$.

We illustrate the above definition with following example

Example 1. *Show that $g(z) = z - \frac{z^2}{2}$ then, $g(z) \in N_{\lambda,4}(e)\lambda \geq 4$. Where e is the identity function. As $g(z) = z - \frac{z^2}{2} = z - \sum_{k=2}^{\infty} b_k z^k$. Therefore $b_2 = \frac{1}{2}$ and $b_k = 0, k \geq 3$. $e = z$. $g(z_1) = g(z_2) \Rightarrow (z_1 - \frac{z_1^2}{2}) = (z_2 - \frac{z_2^2}{2}), z_1, z_2 \in \Delta$.*

Therefore, g is one to one on unit disc Δ and analytic.

$$N_{\gamma,4}(e) = \left\{ g(z) \in N : \sum_{k=2}^{\infty} k\,|pb_k| \leq \gamma \right\}$$

$$\sum_{k=2}^{\infty} k|pb_k| = 2.4.\frac{1}{2} = 4 \leq \gamma.$$

$$\Rightarrow g(z) \in N_{\gamma,4}(e).$$

Example 2. *Show that* $g(z) = z - \frac{z^2}{2}$*, then,* $g(z) \in N_{k,1,1}(e)$*, where* e *is the identity function. As we discussed in Example 1,* $g(z)$ *is one to one in unit disc and*

$$N_{k,1,1}(e) = \left\{ g(z) \in N | g(z) = z - \sum_{k=2}^{\infty} \frac{b_k}{k} z^k \quad and \quad \sum_{k=2}^{\infty} k|\frac{b_k}{k}| \leq 1 \right\}$$

$$= \left\{ g(z) \in N | g(z) = z - \sum_{k=2}^{\infty} \frac{b_k}{k} z^k \quad and \quad \sum_{k=2}^{\infty} |b_k| \leq 1 \right\}$$

$$g(z) = z - \frac{z^2}{2} = z - b_2 z^2.$$

$$\sum_{k=2}^{\infty} |b_k| = \frac{1}{2} \leq 1.$$

Therefore, $g(z) \in N_{k,1,1}(e)$.

Definition 2.4. *The function* $f(z)$ *defined by* (1.1) *is said to be belongs to class* $DA(\eta, \xi, \alpha, \beta, \partial, \lambda, \mathcal{C})$ *iff their exist function* $g \in DA(\eta, \xi, \alpha, \beta, \partial, \lambda)$ *such that*

$$\left| \frac{f(z) - p.g(z)}{g(z)} \right| \leq 1 - \mathcal{C}, \quad where \quad z \in \Delta, 0 \leq c \leq 1.$$

3. Main Results

In the given section, first theorem gives inclusion property related with neighbourhood. Other results provide examples related to the class $DA(\eta, \xi, \alpha, \beta, \partial, \lambda)$.

Theorem 3.1. *Let* $DA(\eta, \xi, \alpha, \beta, \partial, \lambda)$ *and* $\mathcal{C} = 1 - \frac{\gamma}{2}.d$*, where*

$$d = \frac{([1+\partial]^n(1-\lambda) + \lambda\, {}^{n+1}_{\ n}C)(2\xi\beta(2-\alpha) + (1-\beta)}{([1+\partial]^n(1-\lambda) + \lambda\, {}^{n+1}_{\ n}C)(2\xi\beta(2-\alpha) + (1-\beta) - 2\xi\beta(1-\alpha))}.$$

Then $N_{\gamma,p}(DA(\eta, \xi, \alpha, \beta, \partial, \lambda)) \subseteq DA(\eta, \xi, \alpha, \beta, \partial, \lambda, \mathcal{C})$.

Proof. Suppose $g(z) = z - \sum_{k=2}^{\infty} a_k z^k \in DA(\eta, \xi, \alpha, \beta, \partial, \lambda)$. Let

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in N_{\gamma,p} DA(\eta, \xi, \alpha, \beta, \partial, \lambda).$$

Then

$$\sum_{k=2}^{\infty} k|a_k - pb_k| \leq \gamma \Rightarrow \sum_{k=2}^{\infty} |a_k - pb_k| \leq \frac{\gamma}{2}.$$

As $g(z) \in DA(\eta, \xi, \alpha, \beta, \partial, \lambda)$,

$$\sum_{k=2}^{\infty} b_k \leq \frac{2\xi\beta(1-\alpha)}{([1+\partial]^n(1-\lambda) + \lambda\, {}_{n}^{n+1}C)(2\xi\beta(2-\alpha) + (1-\beta))}. \tag{3.1}$$

$$\begin{aligned}
|f(z) - p.g(z)| &= \Big| z - \sum_{k=2}^{\infty} a_k z^k - p.(z - \sum_{k=2}^{\infty} b_k z^k) \Big| \\
&= \Big| \sum_{k=2}^{\infty} (a_k - pb_k) z^k \Big| \\
\Big| \frac{f(z) - p.g(z)}{g(z)} \Big| &= \Big| \frac{\sum_{k=2}^{\infty} (a_k - pb_k) z^k}{z - \sum_{k=2}^{\infty} a_k z^k} \Big| \\
&= \Big| \frac{\sum_{k=2}^{\infty} (a_k - pb_k) z^{k-1}}{1 - \sum_{k=2}^{\infty} b_k z^{k-1}} \Big| \\
&= \Big| \frac{\sum_{k=2}^{\infty} (a_k - pb_k)}{1 - \sum_{k=2}^{\infty} b_k} \Big|.
\end{aligned}$$

Hence, from (3.1)

$$\begin{aligned}
\sum_{k=2}^{\infty} b_k &\geq 1 - \frac{2\xi\beta(1-\alpha)}{([1+\partial]^n(1-\lambda) + \lambda\, {}_{n}^{n+1}C)(2\xi\beta(2-\alpha) + (1-\beta))} \\
&\geq \frac{([1+\partial]^n(1-\lambda) + \lambda\, {}_{n}^{n+1}C)(2\xi\beta(2-\alpha) + (1-\beta)) - 2\xi\beta(1-\alpha)}{([1+\partial]^n(1-\lambda) + \lambda\, {}_{n}^{n+1}C)(2\xi\beta(2-\alpha) + (1-\beta))}.
\end{aligned}$$

$$\begin{aligned}
\Big| (\frac{f(z) - p.g(z)}{g(z)} \Big| &\leq \Big| \frac{\sum_{k=2}^{\infty} (a_k - pb_k)}{1 - \sum_{k=2}^{\infty} b_k} \Big| \\
&\leq \frac{\gamma}{2} \frac{([1+\partial]^n(1-\lambda) + \lambda\, {}_{n}^{n+1}C)(2\xi\beta(2-\alpha) + (1-\beta)}{([1+\partial]^n(1-\lambda) + \lambda\, {}_{n}^{n+1}C)(2\xi\beta(2-\alpha) + (1-\beta)) - 2\xi\beta(1-\alpha))} \\
&\leq \frac{\gamma}{2} d \\
&= 1 - \mathcal{C}.
\end{aligned}$$

Hence, $f \in DA(\eta, \xi, \alpha, \beta, \partial, \lambda, \mathcal{C})$. Therefore,

$$N_{\gamma,p}(DA(\eta, \xi, \alpha, \beta, \partial, \lambda)) \subseteq DA(\eta, \xi, \alpha, \beta, \partial, \lambda, \mathcal{C}).$$

□

Theorem 3.2. $DA(\eta, \xi, \alpha, \beta, \partial, \lambda) \subseteq N_{k,\gamma,1}(e)$. *Where*

$$\gamma = \frac{2\xi\beta(1-\alpha)}{([1+\partial]^n(1-\lambda) + \lambda\, {}^{n+1}_{n}C)(2\xi\beta(2-\alpha) + (1-\beta))},$$

and e *is the identity function.*

Proof. Assume $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in DA(\eta, \xi, \alpha, \beta, \partial, \lambda)$. Then,

$$\begin{aligned}
&\sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda) + \lambda\, {}^{n+k-1}_{n}C)(2\xi\beta(k-\alpha) + (k-1)(1-\beta))a_k \\
&\quad < 2\xi\beta(1-\alpha)([1+\partial]^n(1-\lambda) + \lambda\, {}^{n+1}_{n}C)(2\xi\beta(2-\alpha) + (1-\beta))\sum_{k=2}^{\infty} a_k \\
&\quad \leq \sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda) + \lambda\, {}^{n+k-1}_{n}C)(2\xi\beta(k-\alpha) + (k-1)(1-\beta))a_k \\
&\quad \leq 2\xi\beta(1-\alpha).
\end{aligned}$$

and

$$\sum_{k=2}^{\infty} a_k \leq \frac{2\xi\beta(1-\alpha)}{([1+\partial]^n(1-\lambda) + \lambda\, {}^{n+1}_{n}C)(2\xi\beta(2-\alpha) + (1-\beta))} \tag{3.2}$$

$$N_{k,\gamma,1}f(z) = \left\{ g(z) \in N | g(z) = z - \sum_{k=2}^{\infty} \frac{b_k}{k} z^k \quad \text{and} \quad \sum_{k=2}^{\infty} k|a_k - \frac{b_k}{k}| \leq \gamma \right\}$$

$$N_{k,\gamma,1}(e) = \left\{ g(z) \in N | g(z) = z - \sum_{k=2}^{\infty} \frac{b_k}{k} z^k \quad \text{and} \quad \sum_{k=2}^{\infty} k|\frac{b_k}{k}| \leq \gamma \right\}$$

$$N_{k,\gamma,1}(e) = \left\{ g(z) \in N | g(z) = z - \sum_{k=2}^{\infty} \frac{b_k}{k} z^k \quad \text{and} \quad \sum_{k=2}^{\infty} |b_k| \leq \gamma \right\}. \tag{3.3}$$

Hence, from (3.2) and (3.3)

$$f(z) \in DA(\eta, \xi, \alpha, \beta, \partial, \lambda).$$

Therefore, $DA(\eta, \xi, \alpha, \beta, \partial, \lambda) \subseteq N_{k,\gamma,1}(e)$. □

Theorem 3.3. *Let $f \in DA(\eta, \xi, \alpha, \beta, \partial, \lambda)$. For every $t \geq 0$, let the function $H_t(z)$ be defined by*

$$H_t(z) = (1-t)f(z) + t\int_0^z \frac{f(s)}{s}ds.$$

Then, for

$$M = \left\{H_t | t \geq 0, t \in \mathbb{R}^+\right\}.$$

$M \subseteq DA(\eta, \xi, \alpha, \beta, \partial, \lambda)$.

Proof. Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in DA(\eta, \xi, \alpha, \beta, \partial, \lambda)$. Therefore, from remark 2.1 we have

$$\begin{aligned}
&\sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda) + \lambda\, {}_{n}^{n+k-1}C)(2\xi\beta(k-\alpha) + (k-1)(1-\beta))a_k \\
&\qquad < 2\xi\beta(1-\alpha) \\
H_t(z) &= (1-t)f(z) + t\int_0^z (f(s))/s ds \\
&= (1-t)(z - \sum_{k=2}^{\infty} a_k z^k) + t\int_0^z \frac{s - \sum_{k=2}^{\infty} a_k s^k}{s} ds \\
&= z - tz - \sum_{k=2}^{\infty} a_k z^k + \sum_{k=2}^{\infty} t a_k z^k + tz - t\sum_{k=2}^{\infty} a_k \frac{z^k}{k} \\
&= z - \sum_{k=2}^{\infty}(1 - t + \frac{t}{k})a_k z^k = z - \sum_{k=2}^{\infty} b_k z^k.
\end{aligned}$$

But $1 - t + \frac{t}{k} < 1$ for $k \geq 2$. Therefore,

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{([1+(k-1)\partial]^n(1-\lambda) + \lambda\, {}_{n}^{+k-1}C)}{2\xi\beta(1-\alpha)} \\
&\quad \cdot (2\xi\beta(k-\alpha) + (k-1)(1-\beta)(1 - t + \frac{t}{k})a_k) \\
&< \sum_{k=2}^{\infty} \frac{([1+(k-1)\partial]^n(1-\lambda) + \lambda\, {}_{n}^{n+k-1}C)}{2\xi\beta(1-\alpha)} \\
&\quad \cdot (2\xi\beta(k-\alpha) + (k-1)(1-\beta)a_k) < 1.
\end{aligned}$$

Hence,

$$\sum_{k=2}^{\infty} \frac{([1+(k-1)\partial]^n(1-\lambda)+\lambda\, {}^{n+k-1}_{n}C)}{2\xi\beta(1-\alpha)}$$
$$\cdot\,(2\xi\beta(k-\alpha)+(k-1)(1-\beta)(1-t+\frac{t}{k})a_k) < 1.$$

Therefore, by Remark 2.1,

$$H_t(z) \in DA(\eta,\xi,\alpha,\beta,\partial,\lambda),$$
$$M \subseteq DA(\eta,\xi,\alpha,\beta,\partial,\lambda).$$

□

Theorem 3.4. *Let* $f \in DA(\eta,\xi,\alpha,\beta,\partial,\lambda)$*. Defined the function* $S_t(z)$ *as bellow,*

$$S_t(z) = (1-t)z + t\int_0^z \frac{f(s)}{s}ds, \quad 0 \le t \le 2, \quad z \in \Delta,$$
$$L = \left\{S_t | 0 \le t \le 2, t \in \mathbb{R}^+\right\},$$

Then $L \subseteq DA(\eta,\xi,\alpha,\beta,\partial,\lambda)$*.*

Proof. Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in DA(\eta,\xi,\alpha,\beta,\partial,\lambda)$. Therefore, by Remark 2.1

$$\sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda)+\lambda\, {}^{n+k-1}_{n}C)(2\xi\beta(k-\alpha)+(k-1)(1-\beta))a_k$$
$$< 2\xi\beta(1-\alpha)$$

and

$$S_t(z) = (1-t)z + t\int_0^z \frac{f(s)}{s}ds = z - tz + t\int_0^z \frac{s-\sum_{k=2}^{\infty} a_k s^k}{s}ds$$
$$= z - \sum_{k=2}^{\infty}\frac{t}{k}z^k a_k = z - \sum_{k=2}^{\infty} b_k z^k.$$

Now we will show that $S_t(z) \in DA(\eta,\xi,\alpha,\beta,\partial,\lambda)$. Since $0 \le t \le 2, t \le 1$,

$$\sum_{k=2}^{\infty} \frac{([1+(k-1)\partial]^n(1-\lambda)+\lambda\, {}^{n+k-1}_{n}C(2\xi\beta(k-\alpha)+(k-1)(1-\beta)a_k)}{2\xi\beta(1-\alpha)}\frac{t}{k}$$
$$\le \sum_{k=2}^{\infty} \frac{([1+(k-1)\partial]^n(1-\lambda)+\lambda\, {}^{n+k-1}_{n}C)(2\xi\beta(k-\alpha)+(k-1)(1-\beta)a_k)}{2\xi\beta(1-\alpha)}\frac{t}{2}$$
$$\le 1.$$

Hence, $S_t(z) \in DA(\eta, \xi, \alpha, \beta, \partial, \lambda)$. Therefore, $L \subseteq DA(\eta, \xi, \alpha, \beta, \partial, \lambda)$.

□

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