ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.4, 1973–1981
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.4.56 Spec. Issue on NCFCTA-2020

G-GRACEFUL LABELING OF GRAPHS

AJENDRA KUMAR, VIPIN KUMAR[1], KAMESH KUMAR, PRATIK GUPTA,
AND YOGESH KHANDELWAL

ABSTRACT. After a long period of scramble over analysis and investigations, the notion of graceful labeling came into existence. A mapping f for a graph $G = (R, S)$ is said to be graceful if there exists a bijective mapping $f : R(G) \to N \cup \{0\}$ such that each edge has an induced label

$$\omega(f,\ R(G)) = \{|f(u) - f(v)| : u,\ v \,\epsilon\, R(G)\}$$

and the resulting edge labels are distinct. In this paper, we introduce a new type of graph labeling for a graph $G = (R, S)$ which we call G-graceful labeling. The G-graceful labeling for the graph $G = (R, S)$ with r vertices and $s$edges is an injective function $\rho : R(G) \to \{0, 1, 2, 3, \ldots, t-1\}$such that the induced function $\rho^* : S(G) \to N$ is given by$\rho^*(r, s) = \{\rho^*(r) + \rho^*(s)\}$,the resulting edge label are distinct. In this paper, we also prove that the graphs: path, ladder graph, flower graph, complete bipartite graph and star graph admit G-graceful labeling.

1. INTRODUCTION

In graph theory, there are two important parameters by which a graph $G = (R, S)$ can be represented abstractly and these are vertices and edges. Labeling of graph G means to give labels or weights to the vertices or edges or both

[1]*corresponding author*

2010 *Mathematics Subject Classification.* 05C78.
Key words and phrases. G-graceful graph, path, ladder graph, flower graph, complete bipartite graph, star graph.

under certain conditions and the graph becomes more powerful than its unlabeled structure. After a long period of scramble over analysis and investigations, the notion of graceful labeling came into existence by the effort of Rosa [7] and Golomb [3] and almost all type of graph labeling techniques traces their origin from these papers. A mapping f for a graph $G = (R, S)$ is said to be graceful if there exists a bijective mapping $f : R(G) \to N \cup \{0\}$ such that each edge has an induced label $\omega(f,\ R(G)) = \{|f(u) - f(v)| : u,\ v \epsilon R(G)\}$ and the resulting edge labels are distinct. All types of graph labeling is written in very unique book by Gallian [2] which is considered to the most important book of this area. All graphs considered in this paper are finite, simple, connected and undirected. On the basis of pre-defined graph labeling technique discussed in [2], we introduce a new concept of labeling known as G-graceful labeling and other information which are used for the present investigations are given [1,5,6,8].

2. Definitions

Definition 2.1. *(Ladder Graph) The ladder $L_t(t \geq 2)$ is the product of $P_2 \times P_t$ which contain $2t$ vertices and $3t - 2$ edges [1, 4].*

Definition 2.2. *(Flower Graph) Let $G(R, S)$ be a graph of order t and size ($t - 1$) such that exactly one vertex is adjacent to every other ($t - 1$) vertex. The resulting graph is flower graph with ($t - 1$) petal [1, 4].*

Definition 2.3. *(Complete Bipartite Graph) A complete bipartite graph is a bipartite graph G in which each vertex in R_1 is connected to every vertex inR_2.If$|R_1| = 2\, and\ |R_2| = t$, then complete bipartite graph is written by $K_{2,t}$ [1, 4].*

Definition 2.4. *(Star Graph) Any complete bipartite graph $K_{m,t}$ represent a star graph if $m = 1$ and is denoted by $K_{1,t}$ [1, 4].*

Definition 2.5. *(Path) In m-distant tree if$m = 0$, then 0-distant tree is called a path and is denoted by P_t [1, 4].*

3. Results

In this section, we introduce a new type of graph labeling for a graph $G = (R, S)$ which we call G-graceful labeling.

Definition 3.1. *(G-Graceful Labeling) A function ρ is called G-graceful labeling of a graph $G(R, S)$if $\rho : R(G) \to \{0,\ 1,\ 2,\ 3,\ \ldots,\ t-1\}$ is injective and the induced function $\rho^* : S(G) \to N$ is defined as $\rho^*(t = rs) = \{\rho^*(r) + \rho^*(s)\}$ then edge labels are distinct.*

Theorem 3.1. *Path$(P_t)$is G-graceful graph.*

Proof. Let P_t be a path with vertex set $\{r_1, r_2, r_3, \ldots, r_t\}$ and edge set

$$\{s_1, s_2, s_3, \ldots, s_t\}.$$

We defined a vertex labeling $\rho : R(G) \to \{0, 1, 2, 3, \ldots, t-1\}$ such that

$$\rho(r_1) = 0$$
$$\rho(r_2) = 1$$
$$\rho(r_3) = 2$$
$$\rho(r_4) = 3$$
$$\vdots$$
$$\rho(r_t) = t-1,$$

Vertex labeling can be done in both directions.
The edge labeling function ρ^* is defined as follows $\rho^* : S(G) \to N$ is defined by

$$\rho^*(rs) = \{\rho^*(r) + \rho^*(s)\}$$
$$\rho^*(r_1 r_t) = \{\rho^*(r_1) + \rho^*(r_t)\}$$
$$\vdots$$
$$= \{1, 3, 5, 7, \ldots, 2t-1\}$$

Such that the edge label are district and in increasing order with arithmetic progression whose first term $a = 1$ and$d = 2$. In view of above labeling pattern the path are G-graceful labeling. Hence P_t is a G-graceful graph. □

Illustration: G-graceful labeling of graph P_4 is shown in figure 1.

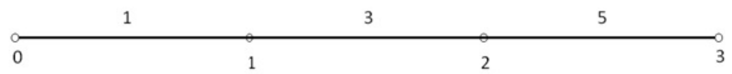


FIGURE 1. G-graceful labeling of the path P_4

Theorem 3.2. *Flower graph is G-graceful.*

Proof. Let $G(R, S)$ be a flower graph, then G has t vertices and $t-1$ edges. Therefore vertex set $R = \{r_1, r_2, r_3, \ldots, r_t\}$ and edge set$S = \{s_1, s_2, s_3, \ldots, s_t\}$. We defined the vertex labeling $\rho : R(G) \to \{0, 1, 2, 3, \ldots, t-1\}$ such that

$$\rho(r_1) = 0$$
$$\rho(r_2) = 1$$
$$\rho(r_3) = 2$$
$$\rho(r_4) = 3$$
$$\vdots$$
$$\rho(r_t) = t-1$$

Such that labeling of the vertices may be clockwise or anticlockwise.
The edge labeling function ρ^* is defined as follows $\rho^* : S(G) \to N$ is defined by

$$\rho^*(rs) = \{\rho^*(r) + \rho^*(s)\}$$
$$\rho^*(r_1 r_t) = \{\rho^*(r_1) + \rho^*(r_t)\}$$
$$\vdots$$
$$= \{1, 2, 3, 4, 5, \ldots, t\}.$$

Then the edge labels are distinct and are in increasing order with arithmetic progression whose first term $a = 1$ and common difference $d = 1$. In view of above labeling pattern the flower graph are G-graceful labeling. Hence all flower graphs are G-graceful graph.

□

Theorem 3.3. *Star graph $K_{1,t}$ is G-graceful graph.*

Illustration: Flower graph with 7 petals in Figure 2.

Proof. Let G=$K_{1,t}$ be a star graph. Let $\{r_1, r_2, r_3, \ldots, r_t\}$ be the vertex set of star graph and it has $1 \times t$ number of edges. We defined the vertex labeling $\rho : R(G) \to \{0, 1, 2, 3, \ldots, t-1\}$ such that

$$\rho(r_1) = 0$$
$$\rho(r_2) = 1$$
$$\rho(r_3) = 2$$

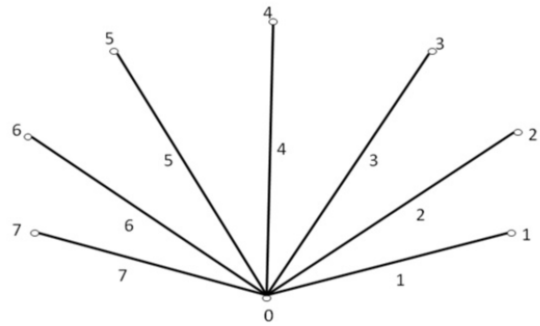


FIGURE 2. G-graceful labeling of flower graph

$$\rho(r_4) = 3$$

$$\vdots$$

$$\rho(r_t) = t - 1$$

The edge label function ρ^* is defined as follows $\rho^* : S(G) \to N$ is defined by

$$\rho^*(rs) = \{\rho^*(r) + \rho^*(s)\}$$

$$\rho^*(r_1 r_t) = \{\rho^*(r_1) + \rho^*(r_t)\}$$

$$\vdots$$

$$= \{1, 2, 3, 4, 5, \ldots, t\}$$

Then the edge labels are distinct and are in increasing order with arithmetic progression whose first term $a = 1$and common difference$d = 1$. In view of above labeling pattern the star graphs are G-graceful labeling. Hence $K_{1,t}$is G-graceful graph. □

Illustration: G-graceful labeling of the graph $K_{1,5}$ is shown in figure 3.

Theorem 3.4. *All complete bipartite graphs*$(K_{2,t})$ *are G-graceful graph.*

Proof. Let $G = K_{2,t}$ be a complete bipartite graph. Let the vertex set be $\{r_1, r_2, r_3, \ldots, r_t, r_{t+1}, r_{t+2}\}$ and $K_{2,t}$ has $2 \times t$ number of edges. The vertex set is

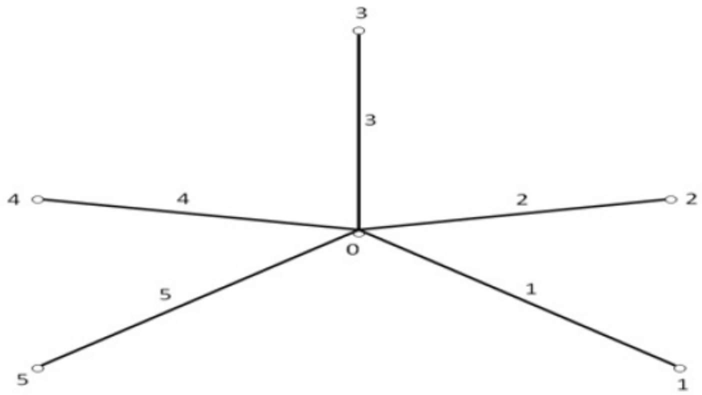


FIGURE 3. G-graceful labeling of the star graph$K_{1,5}$

divided into two set $R_1 \& R_2$where$R_1 = \{r_1, r_2\}$ & $R_2 = \{r_3, r_4, \ldots, r_t, r_{t+1}, r_{t+2}\}$. Define $\rho : R(G) \to \{0, 1, 2, 3, \ldots, t-1\}$ such that

$$\begin{aligned} \rho(r_1) &= 0 \\ \rho(r_2) &= 1 \\ \rho(r_3) &= 2 \\ \rho(r_4) &= 3 \\ &\cdots \\ \rho(r_t) &= t-1, \end{aligned}$$

Label the vertex in both direction and vertex is fixed in top of the graph continuing in this fashion until all the vertices are labeled.
The edge labeling function ρ^* is defined as follows $\rho^* : S(G) \to N$ is defined by

$$\begin{aligned} \rho^*(rs) &= \{\rho^*(r) + \rho^*(s)\} \\ \rho^*(r_1 r_t) &= \{\rho^*(r_1) + \rho^*(r_t)\} \\ &\cdots \\ &= \{1, 2, 3, \ldots, t-1\} \end{aligned}$$

Then the edge labels are distinct. In complete bipartite graph $K_{2,t}$ it is noted that the edge label are in a sequence of natural number except one term are missed i.e. when $t = 1$ then 2^{nd} term are missed in the sequence of natural number, when $t = 2$ then 3^{rd} term are missed, when $t = 3$ then 4rth term are missed, when $t = 4$ then 5^{th} term are missed, continuing like this when $t = k$

then$k+1$ term are missed of the sequence or in other word we can say that when we increase the value of t then the next term are missed in the given sequence of natural number. Hence $K_{2,t}$ is a G-graceful graph. □

Illustration: G-graceful labeling of the graph $K_{2,4}$ is shown in figure 4.

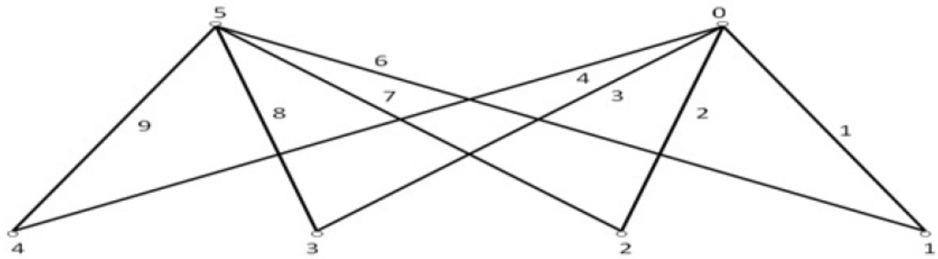
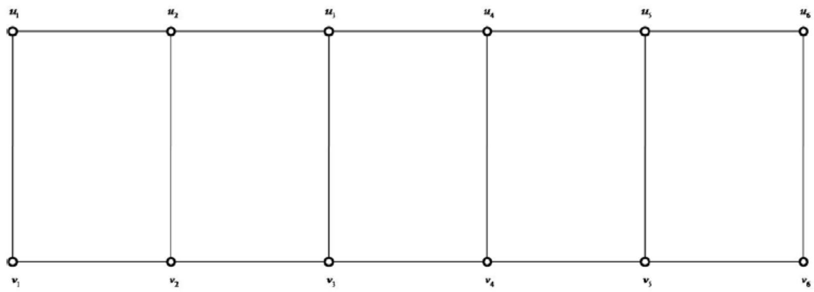


FIGURE 4. G-graceful labeling of the graph $K_{2,4}$

Theorem 3.5. *ladder* (L_t) *are G-graceful graph if* t *is even.*

Proof. let$G = L_t$ be the ladder of $2t$ vertices. Let $\{r_1, r_2, r_3, \ldots, r_t\}$ be the vertices of one path and $\{s_1, s_2, s_3, \ldots, s_t\}$ be the vertices of another path. Labels the vertices from left side of one path by $\{k, k+1, k+2, \ldots, k+(t-1)\}$ where $k=0$ and the other vertices of the path again from left side of path by$\{k+t, k+(t+1), k+(t+2), \ldots, k+(2t-1)\}$.



The edge labeling function ρ^* is defined as follows, $\rho^* : S(G) \rightarrow N$ is defined by$\rho^*(rs) = \{\rho^*(r) + \rho^*(s)\}$ such that each label are distinct. In view of the above labeling pattern the ladder are G-graceful labeling. Hence L_t is a G-graceful graph if t is even. □

Illustration: G-graceful labeling of the graph L_4 is shown in figure 5.

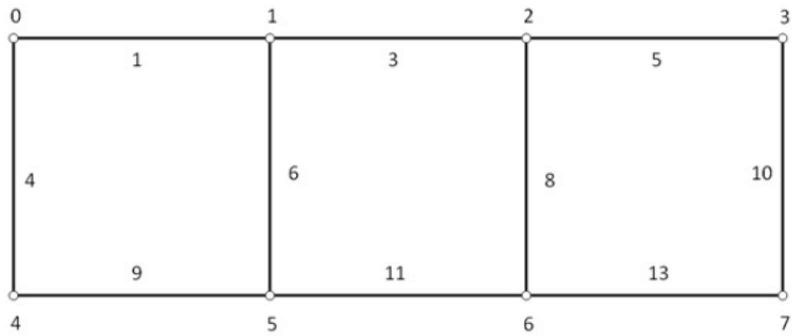


FIGURE 5. G-graceful labeling of L_4

4. CONCLUSION

In this paper we labeled some graphs such as path, ladder graph, flower graph, star graph and complete bipartite graph by proposed G-graceful labeling. The G-graceful labeling reduces the labels of graphs which was higher in other existing labeling of graphs. In future we use this G-graceful labeling to label some other well known existing graphs.

REFERENCES

[1] N. DOE: *Graph Theory with Application to Engineering and Computer Science*, Prentice Hall of India, 2005.

[2] J. A. GALLIAN: *A dynamic survey of graph labeling*, The Electronic Journal of Combinatorics, 2018.

[3] S. W. GOLOMB: *How to Number a Graph: In Graph Theory and Computing*, R. C. Read, ed., Academic press, New York, (1972), 23–37.

[4] F. HARARY: *Graph Theory*, Nervosa Publication House, 2001.

[5] A. KUMAR, D. MISHRA, A. VERMA, V. K. SRIVASTAVA: *Alpha labeling of cyclic graph-1'*, ARS Combinatorial (to appear).

[6] N. MURUGESAN, R. UMA: *Graceful labeling of some graphs and their sub graph*, Asian Journal of Current Engineering and Mathematics, **5** (2012), 1–10.

[7] A. ROSA: *On certain valuation of the vertices of graph*, Theory of Graph (Internat. Symposium, Rome July 1966), Gordon and Breach, N, Y, and Dunned Paris, (1967), 349–355.

[8] V.S. SHIGEHALLI, A. MASARGUPPI: ϕ-graceful labeling of some graphs, Journal of Computer and Mathematical Science **6**(2) (2015), 127–133.

DEPARTMENT OF MATHEMATICS AND STATISTICS
GURUKULA KANGRI VISHWAVIDYALAYA
HARIDWAR-249404, UTTARAKHAND, INDIA
E-mail address: ajendra.gkv@gmail.com

DEPARTMENT OF MATHEMATICS AND STATISTICS
GURUKULA KANGRI VISHWAVIDYALAYA
HARIDWAR-249404, UTTARAKHAND, INDIA
E-mail address: vipinbataar6768@gmail.com

DEPARTMENT OF MATHEMATICS
TEERTHANKER MAHAVEER UNIVERSITY
MORADABAD-244001, UTTAR PRADESH, INDIA
E-mail address: kameshkumar.2012@gmail.com

DEPARTMENT OF MATHEMATICS
MANDSAUR UNIVERSITY
MANDSAUR-458001, MADHYA PRADESH, INDIA
E-mail address: pratikgupta1810@gmail.com

DEPARTMENT OF MATHEMATICS
JAIPUR NATIONAL UNIVERSITY
JAIPUR-302017, RAJASTHAN, INDIA
E-mail address: yogeshmaths81@gmail.com