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ABELIAN-TAUBERIAN THEOREM FOR LAPLACE-STIELTJES TRANSFORM OF HYPERFUNCTIONS

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ABSTRACT. Hyperfunctions are the generalized functions introduced by Mikio Sato in the late 1950's. It can be considered as the analytic equivalent of distributions invented by Schwartz. Here we introduced the combined transform, Laplace-Stieltjes transform to hyperfunctions. Some properties of the combined transform is proved. Abelian - Tauberian theorem is also proved for this combined transform.

1. INTRODUCTION

Mikio Sato established the notion of hyperfunctions to mention his generalisation about the conceptualization of functions. Urs Graf used Sato's idea to generalize the concept of function of a real variable using the classical complex function theory of analytic function and applied various transforms like Laplace transform, Fourier transform, Hilbert transform, Mellin transforms, Hankel transform to a class of hyperfunctions in his book.

In this paper the combined transform, Laplace-Stieltjes transform is applied to a class of hyperfunctions having bounded exponential growth . We have established some properties of this background. Abelian-Tauberian type theorem is also proved for this combined transform of hyperfunctions.

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2. Preliminaries

Denote the complex plane by $\mathfrak{C}$, $\mathfrak{C}_+ = \{\zeta \in \mathfrak{C} : I\zeta > 0\}$, $\mathfrak{C}_- = \{\zeta \in \mathfrak{C} : I\zeta < 0\}$ These sets represents the upper half-plane and lower half-plane of $\mathfrak{C}$ respectively.

Definition 2.1. *[1] $J \subseteq \mathbb{R}$ is open. An open set $\mathfrak{N}(J) \subset \mathfrak{C}$ is called a complex neighborhood of J, if $\mathfrak{N}(J) \setminus J$ is open in $\mathfrak{N}(J)$. Let $\mathfrak{N}_+(J) = \mathfrak{N}(J) \cap \mathfrak{C}_+$ and $\mathfrak{N}_-(J) = \mathfrak{N}(J) \cap \mathfrak{C}_-$, $\mathfrak{O}(\mathfrak{N}(J) \setminus J) =$ Ring of holomophic functions in $\mathfrak{N}(J) \setminus J$ For an interval $J \subseteq \mathbb{R}$, $A(\zeta) \in \mathfrak{O}(\mathfrak{N}(J) \setminus J)$ can be expressed as as*

$$A(\zeta) = \begin{cases} A_+(\zeta) & \textit{for } \zeta \in \mathfrak{N}_+(J), \\ A_-(\zeta) & \textit{for } \zeta \in \mathfrak{N}_-(J), \end{cases}$$

where $A_+(\zeta) \in \mathfrak{O}(\mathfrak{N}_+(J))$ and $A_-(\zeta) \in \mathfrak{O}(\mathfrak{N}_-(J))$ are called upper and lower component of $A(\zeta)$ respectively.

Definition 2.2. *[1] $A(\zeta), B(\zeta) \in \mathfrak{O}(\mathfrak{N}(J) \setminus J)$ are equivalent if for $\zeta \in \mathfrak{N}_1(J) \cap \mathfrak{N}_2(J)$, $A(\zeta) = B(\zeta) + \phi(\zeta)$, with $\phi(\zeta) \in \mathfrak{O}(\mathfrak{N}(J))$ where $\mathfrak{N}_1(J)$ and $\mathfrak{N}_2(J)$ are complex neighborhoods of J of $A(\zeta)$ and $B(\zeta)$ respectively. An equivalence class of functions $A(\zeta) \in \mathfrak{O}(\mathfrak{N}(J) \setminus J)$ defines a hyperfunction $a(x)$ on J. It is denoted by $a(x) = [A(\zeta)] = [A_+(\zeta), A_-(\zeta)]$. $A(\zeta)$ is called defining or generating function of the hyperfunction. $\mathfrak{B}(J) =$ Set of all hyperfunctions defined on the real interval $J =\mathfrak{O}(\mathfrak{N}(J) \setminus J) \setminus \mathfrak{O}(\mathfrak{N}(J))$. The value of a hyperfunction $a(x) = [A(\zeta)]$ $a(x) = A(x + i0) - A(x - i0) = \lim_{\epsilon \to 0^+} \{A_+(x + i\epsilon) - A_-(x - i\epsilon)\}$ provided the limit exists.*

Definition 2.3. *The number $\sigma_- = \sigma_-(a)$ is called the growth index of $a(x) \in \mathfrak{O}(\mathbb{R}_+)$. The number $\sigma_+ = \sigma_+(a)$ is called the growth index of $a(x) \in \mathfrak{O}(\mathbb{R}_-)$. The method for finding growth index are described in [1].*

Definition 2.4. *[1] A function or a hyperfunction is said to be of bounded exponential growth, if it is of bounded exponential growth for $x \to -\infty$ as well as for $x \to \infty$.*

Definition 2.5. *[2]* ***Stieltjes Transformation of Hyperfunctions.*** *For a hyperfunction $g(x) = [G(\zeta)]$ of bounded exponential growth the Stieltjes transform is defined by*

$$\tilde{g}(t) = \mathcal{S}[g(x)](t) = \int_0^\infty \frac{g(x)}{x+t} dx = \int_0^\infty \frac{G(\zeta)}{\zeta + t} d\zeta, \textit{ where } t \in \mathbb{R}$$

if it exists.

3. Laplace-Stieltjes Transform of Hyperfunctions

We define Laplace-Stieltjes transform for Hyperfunctions with defining function having bounded variation property.

Definition 3.1. *A hyperfunction* $h(x) = [H(\zeta)] \in \mathfrak{B}(J)$*, where* J *is a closed subset of* $\mathbb{R}$ *is said to be of bounded variation if all the functions* $G(\zeta) \in [H(\zeta)]$ *are of bounded variation (i.e.* $ReG(\zeta)$ *and* $ImG(\zeta)$ *are real functions of bounded variation).*

Definition 3.2. *Let* $h(x) = [H(\zeta)] \in \mathfrak{B}(J)$*, where* $J \subset [0, \infty)$ *is closed , be a measurabe, non decreasing, exponetially bounded real valued hyperfunction of bounded variation. For the complex variable* s *the Laplace-Stieltjes transform is defined as*

$$\mathfrak{L}_{\mathcal{S}}[h(x)](s) = h^*(s) = \int_0^\infty e^{-sx} dh(x) = \int_0^\infty e^{-s\zeta} dH(\zeta),$$

provided the integral converges at some point s_0*. Then it converges* $\forall s$ *with* $\mathcal{R}s > \mathcal{R}s_0$*. (Here the integral is Stieltjes Integral).*

Remark 3.1. *Let* $\mathfrak{B}^{exp}_{m,bv}(J)$ *denotes the set of all measurable, real valued, non decreasing, exponentially bounded hyperfunctions of bounded variation for a closed set* $J \subset \mathbb{R}$*.*

Proposition 3.1. *For* $h(x) \in \mathfrak{B}^{exp}_{m,bv}(J)$*,* $\mathfrak{L}_{\mathcal{S}}[h(x)](s)$ *is strictly positive.*

Proof. Let $h(x) > 0$. Then since the integral of a non negative function is non negative, $\mathfrak{L}_{\mathcal{S}}[h(x)](s) > 0$. □

Example 1. *Consider the hyperfunction* $h(x) = x^n$*. Here the defining function is* $H(\zeta) = \zeta^n$ *and*

$$\begin{aligned} h^*(s) &= \mathfrak{L}_{\mathcal{S}}[h(x)](s) = \int_0^\infty e^{-sx} d(x^n) = \int_0^\infty e^{-s\zeta} d(\zeta^n) \\ &= \int_0^\infty e^{-s\zeta} n\zeta^{n-1} d\zeta = \frac{n!}{s^n}. \end{aligned}$$

Remark 3.2. *Next proposition is the relation connecting Laplace-Stieltjes transform and the Laplace transform of a hyperfunction.*

Proposition 3.2. *If* $h(x) = [H(\zeta)] \in \mathfrak{B}^{exp}_{m,bv}(J)$ *with* $\sigma_-(h) < \mathcal{R}(s) < \sigma_+(h)$*, then*

$$\mathfrak{L}_{\mathcal{S}}[h(x)](s) = s\mathfrak{L}[h(x)](s).$$

Proof. Let $h(x) = [H(\zeta)] \in \mathfrak{B}^{exp}_{m,bv}(J)$ with $\sigma_-(h) < \mathcal{R}(s) < \sigma_+(h)$.

$$\begin{aligned}\mathfrak{L}_{\mathcal{S}}[h(x)](s) &= \int_0^\infty e^{-sx}dh(x) = \int_0^\infty e^{-s\zeta}dH(\zeta)\\ &= \int_0^\infty e^{-s\zeta}H^{'}(\zeta)d\zeta = \mathfrak{L}[h^{'}(x)](s) = s\mathfrak{L}[h(x)](s).\end{aligned}$$

□

3.1. **Operational Properties.**

Proposition 3.3. *If* $h(x) = [H(\zeta)], g(x) = [G(\zeta)] \in \mathfrak{B}^{exp}_{m,bv}(J)$ *with Laplace-Stieltjes transforms* $h^*(s)$ *and* $g^*(s)$ *respectively then*

$$\mathfrak{L}_{\mathcal{S}}[h(x) + g(x)](s) = \mathfrak{L}_{\mathcal{S}}[h(x)](s) + \mathfrak{L}_{\mathcal{S}}[g(x)](s).$$

Also for some constant c*,*

$$\mathfrak{L}_{\mathcal{S}}[ch(x)](s) = c\mathfrak{L}_{\mathcal{S}}[h(x)](s).$$

Proposition 3.4. *If* $h(x) = [H(\zeta)] \in \mathfrak{B}^{exp}_{m,bv}(J)$ *and* $\sigma_-(h) < \mathcal{R}(s) < \sigma_+(h)$ *with Laplace-Stieltjes transform* $\mathfrak{L}_{\mathcal{S}}[h(x)](s)$ *then*

$$\mathfrak{L}_{\mathcal{S}}[h^{'}(x)](s) = s\mathfrak{L}_{\mathcal{S}}[h(x)](s) - sh(0) - h^{'}(0).$$

Proof. Let $h(x) = [H(\zeta)] \in \mathfrak{B}^{exp}_{m,bv}(J)$ and $\sigma_-(h) < \mathcal{R}(s) < \sigma_+(h)$ with Laplace-Stieltjes transform $\mathfrak{L}_{\mathcal{S}}[h(x)](s)$. Then

$$\begin{aligned}\mathfrak{L}_{\mathcal{S}}[h^{'}(x)](s) &= \int_0^\infty e^{-sx}d(h^{'}(x)) = \int_0^\infty e^{-s\zeta}d(H^{'}(\zeta)) = \int_0^\infty e^{-s\zeta}H^{''}(\zeta)d\zeta\\ &= \mathfrak{L}[h^{''}(x)](s) = s^2\mathfrak{L}[h(x)](s) - sh(0) - h^{'}(0)\\ &= s\mathfrak{L}_{\mathcal{S}}[h(x)](s) - sh(0) - h^{'}(0).\end{aligned}$$

□

Proposition 3.5. *If* $h(x) = [H(\zeta)] \in \mathfrak{B}^{exp}_{m,bv}(J)$ *and* $\sigma_-(h) < \mathcal{R}(s) < \sigma_+(h)$ *then for any* $a \in \mathfrak{C}$*,*

$$\mathfrak{L}_{\mathcal{S}}[e^{ax}h(x)](s) = \mathfrak{L}[h^{'}(x)](s-a) + a\mathfrak{L}[h(x)](s-a),$$

where $\sigma_-(h) + \mathcal{R}(a) < \mathcal{R}(s) < \sigma_+(h) + \mathcal{R}(a)$*.*

Proof. Let $h(x) = [H(\zeta)] \in \mathfrak{B}^{exp}_{m,bv}(J)$ and $\sigma_-(h) < \mathcal{R}(s) < \sigma_+(h)$. Also let $a \in \mathbb{C}$ be a constant with $\sigma_-(h) + \mathcal{R}(a) < \mathcal{R}(s) < \sigma_+(h) + \mathcal{R}(a)$

$$\begin{aligned}
\mathfrak{L}_{\mathcal{S}}[e^{ax}h(x)](s) &= \int_0^\infty e^{-sx} d(e^{ax}h(x)) = \int_0^\infty e^{-s\zeta} d(e^{a\zeta}H(\zeta)) \\
&= \int_0^\infty e^{-s\zeta}(e^{a\zeta}H'(\zeta) + ae^{a\zeta}H(\zeta))d\zeta \\
&= \int_0^\infty e^{-s\zeta}e^{a\zeta}H'(\zeta)d\zeta + a\int_0^\infty e^{-s\zeta}e^{a\zeta}H(\zeta)d\zeta \\
&= \int_0^\infty e^{-(s-a)\zeta}H'(\zeta)d\zeta + a\int_0^\infty e^{-(s-a)\zeta}H(\zeta)d\zeta \\
&= \mathfrak{L}[h'(x)](s-a) + a\mathfrak{L}[h(x)](s-a).
\end{aligned}$$

□

Proposition 3.6. *If $h(x) = [H(\zeta)] \in \mathfrak{B}^{exp}_{m,bv}(J)$ with Laplace-Stieltjes transform $\mathfrak{L}_{\mathcal{S}}[h(x)](s)$ and $\sigma_-(h) < \mathcal{R}(s) < \sigma_+(h)$ then*

$$\mathfrak{L}_{\mathcal{S}}[h^n(x)](s) = s^n \mathfrak{L}_{\mathcal{S}}[h(x)](s),$$

if the convergence strip is same.

Proof. Let $h(x) = [H(\zeta)] \in \mathfrak{B}^{exp}_{m,bv}(J)$ with Laplace-Stieltjes transform $\mathfrak{L}_{\mathcal{S}}[h(x)](s)$ and $\sigma_-(h) < \mathcal{R}(s) < \sigma_+(h)$. Then

$$\begin{aligned}
\mathfrak{L}_{\mathcal{S}}[h^n(x)](s) &= \int_0^\infty e^{-sx} d(h^n(x)) = \int_0^\infty e^{-s\zeta} d(H^n(\zeta)) \\
&= \int_0^\infty e^{-s\zeta} H^{n+1}(\zeta)d\zeta = \mathfrak{L}[h^{n+1}(x)](s) \\
&= s^{n+1}\mathfrak{L}[h(x)](s) = s^n \mathfrak{L}_{\mathcal{S}}[h(x)](s).
\end{aligned}$$

□

Remark 3.3. *We prove the existence Laplace-Stieltjes transform for convolution of hyperfunctions*

Proposition 3.7. *If $g(x) = [G(\zeta)], h(x) = [H(\zeta)] \in \mathfrak{B}^{exp}_{m,bv}(J)$ having compact support $[a, b]$ and $[c, d]$ respectively with $g^*(s) = \mathfrak{L}_{\mathcal{S}}[g(x)](s)$ and $h^*(s) = \mathfrak{L}_{\mathcal{S}}[h(x)](s)$ exists then the Laplace-Stieltjes transform of the convolution $g(x) * h(x)$ exists and*

$$\mathfrak{L}_{\mathcal{S}}[g(x) * h(x)](s) = \frac{1}{s}\mathfrak{L}_{\mathcal{S}}[g(x)](s)\mathfrak{L}_{\mathcal{S}}[h(x)](s).$$

Proof. Let $g(x) = [G(\zeta)], h(x) = [H(\zeta)] \in \mathfrak{B}^{exp}_{m,bv}(J)$ having compact support $[a, b]$ and $[c, d]$ respectively.

Then the convolution $g(x) * h(x)$ of $g(x)$ and $h(x)$ is exists as a hyperfunction with compact support contained in $[a + c, b + d]$. In general,

$$g(x) * h(x) = \int_0^\infty G(\zeta - \tau)H(\tau)d\tau.$$

Hence,

$$\begin{aligned}
\mathfrak{L}_{\mathcal{S}}[g(x) * h(x)](s) &= \mathfrak{L}_{\mathcal{S}}[\int_0^\infty G(\zeta - \tau)H(\tau)d\tau](s) \\
&= s\mathfrak{L}[\int_0^\infty G(\zeta - \tau)H(\tau)d\tau](s) \\
&= s\int_0^\infty e^{-s\zeta}(\int_0^\infty G(\zeta - \tau)H(\tau)d\tau)d\zeta \\
&= s\int_0^\infty H(\tau)d\tau \int_0^\infty e^{-s\zeta}G(\zeta - \tau)d\zeta.
\end{aligned}$$

Putting $\zeta - \tau = \rho$ we get,

$$\mathfrak{L}_{\mathcal{S}}[g(x) * h(x)](s) = s\int_0^\infty H(\tau)d\tau \int_0^\infty e^{-s(\tau+\rho)}G(\rho)d\rho$$

$$= s\int_0^\infty e^{-s\tau}H(\tau)d\tau \int_0^\infty e^{-s\rho}G(\rho)d\rho = s.\mathfrak{L}[g(x)](s).\mathfrak{L}[h(x)](s)$$

$$= s.\frac{1}{s}\mathfrak{L}_{\mathcal{S}}[g(x)](s).\frac{1}{s}\mathfrak{L}_{\mathcal{S}}[h(x)](s) = \frac{1}{s}\mathfrak{L}_{\mathcal{S}}[g(x)](s)\mathfrak{L}_{\mathcal{S}}[h(x)](s).$$

□

3.2. **Inversion formula for Laplace-Stieltjes Transform of Hyperfunction.** Salltz [3] defined an inversion formula for the Laplace-Stieltjes transform of ordinary functions. The inversion formula for Laplace-Stieltjes transform of a hyperfunction is defined as follows.

Definition 3.3. *The inversion formula for the Laplace-Stieltjes transform of the hyperfunction in* $\mathfrak{B}^{exp}_{m,bv}(J)$ *is defined as*

$$h(x) = \lim_{b\to\infty} \int_{a-ib}^{a+ib} \frac{e^{sx}}{s} h^*(s)ds,$$

where $a > 0$ *is greater than the radius of convergence of the integral.*

4. Abelian-Tauberian Theorem for Laplace-Stieltjes Transform of Hyperfunctions

Let $\mathfrak{B}^{exp+}_{m,bv}(J)$ denote the set of all non-decreasing, non-negative, real valued, holomorphic, measurable, exponentially bounded hyperfunction of bounded variation defined on the closed subset $J \subset [0, \infty)$.

Lemma 4.1. *If $h(x) = [H(\zeta)]$, $g(x) = [G(\zeta)]$ are in $\mathfrak{B}^{exp+}_{m,bv}(J)$ with Laplace-Stieltjes transforms $h^*(s) = \mathfrak{L}_{\mathcal{S}}[h(x)](s)$ and $g^*(s) = \mathfrak{L}_{\mathcal{S}}[g(x)](s)$ respectively, having a common vertical strip of convergence and $h^*(s) = g^*(s)$ then $h(x) = g(x)$.*

Proof. Suppose that $h^*(s) = g^*(s)$. Then,

$$\begin{aligned}
h^*(s) = g^*(s) \quad &\Rightarrow \quad \mathfrak{L}_{\mathcal{S}}[h(x)](s) = \mathfrak{L}_{\mathcal{S}}[g(x)](s) \\
&\Rightarrow \quad \int_0^\infty e^{-sx} dh(x) = \int_0^\infty e^{-sx} dg(x) \\
&\Rightarrow \quad \int_0^\infty e^{-s\zeta} dH(\zeta) = \int_0^\infty e^{-s\zeta} dG(\zeta) \\
&\Rightarrow \quad \int_0^\infty e^{-s\zeta} d(H(\zeta) - G(\zeta)) = 0 \\
&\Rightarrow \quad H(\zeta) = G(\zeta) \\
&\Rightarrow \quad h(x) = g(x)
\end{aligned}$$

□

Theorem 4.1. *Let $(h_n(x)) = ([H_n(\zeta)])$ be a sequence of hyperfunction in $\mathfrak{B}^{exp+}_{m,bv}(J)$ with compact support.*

(a) *Let $h(x) = [H(\zeta)]$ is a measurable hyperfunction having support contained in $(0, \infty)$ such that $h_n(x) \to h(x)$ for all points x at which h_n's and h are holomorphic. If there exists $t \geq 0$ such that $\sup_{n \geq 1} \mathfrak{L}_{\mathcal{S}}[h_n(x)](t) < \infty$ then $\mathfrak{L}_{\mathcal{S}}[h_n(x)](s) \to \mathfrak{L}_{\mathcal{S}}[h(x)](s)$ as $n \to \infty$ for all $s > t$.*

(b) *Suppose there exists $t \geq 0$ such that $\mathfrak{L}_{\mathcal{S}}[h_n(x)](s) \to \mathfrak{L}_{\mathcal{S}}[h(x)](s)$ as $n \to \infty$ for all $s > t$ then $h_n(x) \to h(x)$ for all points x at which h_n's and h are holomorphic, if the Laplace-Stieltjes transforms of h_n's and h have a common vertical strip of convergence.*

Proof.

(a) Let

$$A = \sup_{n \geq 1} \mathfrak{L}_{\mathcal{S}}[h_n(x)](t) < \infty.$$

Then for any $s > t$ and $x \in (0, \infty)$,

$$\int_0^\infty e^{-sx} dh_n(x) \to \int_0^\infty e^{-sx} dh(x),$$

by Dominated convergence theorem for hyperfunctions.

Let $s > t$ and $\epsilon > 0$ such that f is holomorphic at $y \in (0, \infty)$ with $Ae^{-(s-t)y} \leq \epsilon$,

$$\begin{aligned}\int_0^y e^{-sx} dh_n(x) &\leq \mathfrak{L}_{\mathcal{S}}[h_n(x)](s) \leq \int_0^y e^{-sx} dh_n(x) \\ &+ e^{-(s-t)y} \int_y^\infty e^{-tx} dh_n(x) \leq \int_0^y e^{-sx} dh_n(x) + \epsilon.\end{aligned}$$

Then

$$\begin{aligned}\int_0^y e^{-sx} dh(x) &\leq \liminf_{n\to\infty} \mathfrak{L}_{\mathcal{S}}[h_n(x)](s) \leq \limsup_{n\to\infty} \mathfrak{L}_{\mathcal{S}}[h_n(x)](s) \\ &\leq \int_0^y e^{-sx} dh(x) + \epsilon\,.\end{aligned}$$

Letting $y \to \infty$ along holomorphic points of $h(x) = [H(\zeta)]$

$$\begin{aligned}\int_0^\infty e^{-sx} dh(x) &\leq \liminf_{n\to\infty} \mathfrak{L}_{\mathcal{S}}[h_n(x)](s) \leq \limsup_{n\to\infty} \mathfrak{L}_{\mathcal{S}}[h_n(x)](s), \\ &\leq \int_0^\infty e^{-sx} dh(x) + \epsilon\end{aligned}$$

i.e.,

$$\mathfrak{L}_{\mathcal{S}}[h(x)](s) \leq \liminf_{n\to\infty} \mathfrak{L}_{\mathcal{S}}[h_n(x)](s) \leq \limsup_{n\to\infty} \mathfrak{L}_{\mathcal{S}}[h_n(x)](s) \leq \mathfrak{L}_{\mathcal{S}}[h(x)](s) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$\mathfrak{L}_{\mathcal{S}}[h_n(x)](s) \to \mathfrak{L}_{\mathcal{S}}[h(x)](s),$$

as $n \to \infty$ for all $s > t$.

(b) Suppose that

$$\mathfrak{L}_{\mathcal{S}}[h_n(x)](s) \to \mathfrak{L}_{\mathcal{S}}[h(x)](s),$$

as $n \to \infty$ for all $s > t$ and the Laplace-Stieltjes transforms of h_n's and h have a common vertical strip of convergence.

By previous Lemma and dominated convergence theorem for hyperfunctions

$$h_n(x) = \int_0^\infty e^{sx} \mathfrak{L}_{\mathcal{S}}[h_n(x)](s) ds \to \int_0^\infty e^{sx} \mathfrak{L}_{\mathcal{S}}[h(x)](s) ds = h(x).$$

□

Proposition 4.1. *(Abelian theorem for Laplace-Stieltjes Transformation of Hyperfunctions) For* $h(x) \in \mathfrak{B}^{exp+}_{m,bv}(J)$*, if*

$$\lim_{x\to\infty} \frac{h(x)}{x^n} = \frac{M}{n!},$$

then

$$\lim_{s\to 0} s^n h^*(s) = M,$$

where $n \geq 0$ *is a number and* M *is a constant.*

Proof. Let $h(x) \in \mathfrak{B}^{exp+}_{m,bv}(J)$, $n \geq 0$ and M be a constant. Suppose that

$$\lim_{x\to\infty} \frac{h(x)}{x^n} = \frac{M}{n!}.$$

Then

$$h(x) \to \frac{Mx^n}{n!} \;\; as \;\; x \to \infty.$$

Hence by the previous proposition 4.1.

$$h^*(s) \to \frac{M}{n!}.\frac{n!}{s^n} \;\; as \;\; s \to 0;$$

$$i.e. s^n h^*(s) \to M \;\; as \;\; s \to 0.$$

Hence $\lim_{s\to 0} s^n h^*(s) = M$. □

Proposition 4.2. *(Tauberian theorem for Laplace-Stieltjes Transformation of Hyperfunctions) Let* $h(x) \in \mathfrak{B}^{exp+}_{m,bv}(J)$ *with Laplace-Stieltjes transform,* $h^*(s) = \int_0^\infty e^{-sx} dh(x)$*, which converges for some* $\mathcal{R}(s) > 0$ *and* $\lim_{s\to 0} s^n h^*(s) = M$*, for some constant* M *and* $n > 0$*. Then*

$$\lim_{x\to\infty} \frac{h(x)}{x^n} = \frac{M}{n!}.$$

Proof. Let $h(x) \in \mathfrak{B}^{exp+}_{m,bv}(J)$ with $h^*(s) = \int_0^\infty e^{-sx} dh(x)$, which converges for some $\mathcal{R}(s) > 0$ and $\lim_{s\to 0} s^n h^*(s) = M$, for some constant M and $n > 0$,

$$\text{i.e.} \quad s^n h^*(s) \to M \;\; as \;\; s \to 0,$$

$$\text{i.e.} \quad \frac{h^*(s)}{M} \to \frac{1}{s^n} \;\; as \;\; s \to 0.$$

Then by proposition 4.1 we have

$$\frac{h(x)}{M} \to \frac{x^n}{n!} \;\; as \;\; x \to \infty,$$

$$\frac{h(x)}{x^n} \to \frac{M}{n!} \;\; as \;\; x \to \infty.$$

Hence,

$$\lim_{x\to\infty} \frac{h(x)}{x^n} = \frac{M}{n!}.$$

□

Here we established some operational properties of Laplace- Stieltjes transform of a subclass of the linear space of of hyperfunctions. Also proved Abelian-Tauberian theorems, which helps to investigate the convergence of the Laplace-Stieltjes transform integral.

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