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IDEALS AND CONGRUENCE WITH RESPECT TO A FRAME HOMOMORPHISM

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ABSTRACT. In the background of point free topology, given a frame L and a frame homomorphism $f : L \to L$, for each $b \in L$, ideals of the form $\langle f \rangle_b = \{a \in L : \Sigma_{f(a)} \subseteq \Sigma_b\}$ is constructed and its properties are studied. These ideals are utilized to form a frame congruence on L and hence a sublocale of L. By assigning the proper order, the set $J_f = \{\langle f \rangle_b : b \in L\}$ has been shaped as a locale of ideals of L. An equivalent condition for the locale J_f to be compact is also derived.

1. INTRODUCTION

Topology in a topological space can be viewed as a lattice. In the same way, from a complete lattice having additional property, a topological space can be constructed. Marshall Stone [8] related the concept of lattice theory with that of topology. Banaschewski [1], [2], John Isabell [5], Picado, Pult [7], Johnstone [6] etc continued the work and most of the topological concepts get extended into the background pointfree topology.

In this study, we have constructed ideals $\langle f \rangle_b = \{a \in L : \Sigma_{f(a)} \subseteq \Sigma_b\}$ from a fiven frame homomorphism $f : L \to L$. The necessry condition for these ideals to be prime is also investigated. The ideals of the form $\langle f \rangle_b$ have been utilized to define a congruence relation on the frame L and hence a sublocale of L. Using

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the ideals $\langle f\rangle_b$, a locale of the form $J_f = \{\langle f\rangle_b : b \in L\}$ has build up and some of its properties are investigated.

Throughout this paper $\Omega(X)$ denotes the topology on the topological space $(X, \Omega(X))$ and L is a locale.

2. Preliminaries

Definition 2.1. *[7] A frame (or a locale) is a complete lattice F satisfying the infinite distributivity law $x \wedge \bigvee Y = \bigvee\{x \wedge y; y \in Y\}$ for all $x \in F$ and $Y \subseteq F$.*

Definition 2.2. *[7] If $L, M,$ are frames,a map $h : L \to M$ is said to be frame homomorphism if it preserves all finite meets (including the top 1) and all joins (including the bottom 0).*

Example 1. *[7]*

(i) *If (X, τ) is a topological space, τ is a frame.*
(ii) *Every finite lattice having distributive property is a frame.*

Definition 2.3. *[6] A non empty subjoin semilattice I of a locale L is an ideal, if it is a lower set.*

Definition 2.4. *[6] A proper ideal I is prime if $x \wedge y \in I$ implies that either $x \in I$ or $y \in I$.*

Recall that a flter F is an upper set closed under finite meets and that it is *prime* resp. *completely prime* if $a_1 \vee a_2 \in F$ resp. $\bigvee_i a_i \in F$ implies that $a_i \in F$ for some i. Completely prime filters are denoted by c.p filters [7].

Example 2. *[7] Neighbourhood filters are completely prime in the frame $\Omega(X)$.*

For $a \in L$, set $\Sigma_a = \{F \subseteq L; F \neq \phi, F \;\; is \;\; c.p \;\; filters \;\; ; a \in F\}$. Then we have $\Sigma_0 = \phi$, $\Sigma_{\bigvee a_i}$=$\bigcup \Sigma_{a_i}$, $\Sigma_{a\wedge b} = \Sigma_a \cap \Sigma_b$ and $\Sigma_1 = \{all \;\; c.p \;\; filters\}$.
From Definition of c.p filters, if $a \leq b$, then $\Sigma_a \subseteq \Sigma_b$. But $\Sigma_a \subseteq \Sigma_b$ need not imply $a \leq b$.

Definition 2.5. *[7] The spectrum of a frame is defined as follows:*

$$Sp(L) = (\{\text{all completely prime filters in } L\}, \quad \{\Sigma_a : a \in L\}).$$

Then Sp(L)is a topological space with the topology $\Omega(Sp(L)) = \{\Sigma_a : a \in L\}$. Topological space associated with the frame L is $Sp(L)$.

Definition 2.6. *[7] A subset $A \subseteq L$ is said to be a cover of L if $\bigvee A = 1$. $B \subseteq A$ is a sub cover if $\bigvee B = 1$. A compact frame L is a frame having the property that every cover of L has a finite sub cover.*

Definition 2.7. *[7] Let L be a frame. A relation θ on L is a congruence relation on L if*

(i) *θ is an equivalence relation*
(ii) $(a, b) \in \theta \Rightarrow (a \wedge c, b \wedge c) \in \theta$ *and* $(a \vee \bigvee S, b \vee \bigvee S) \in \theta$

Definition 2.8. *[7] Let L be a lattice.$p \neq 1 \in L$ is meet irreducible if $x \wedge y \leq p$ implies that either $x \leq p$ or $y \leq p$ for all $x, y \in L$.*

3. IDEALS FROM FRAME HOMOMORPHISM

Consider the morphism $f : L \to L$ in the category **Frm**. For any $b \in L$, set $\langle f \rangle_b = \{a \in L : \Sigma_{f(a)} \subseteq \Sigma_b\}$. Since $f(0) = 0$, we have $\Sigma_{f(0)} = \phi$. Thus $0 \in \langle f \rangle_b$ and hence $\langle f \rangle_b$ is non empty.

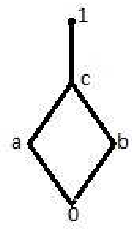
Lemma 3.1. *For any $b \in L$,$\langle f \rangle_b$ is an ideal on L. If Σ_b is a meet irreducible element in $\Omega(Sp(L))$, then the ideal $\langle f \rangle_b$ is a prime.*

Proof. Choose $a, c \in \langle f \rangle_b$. Then we have $\Sigma_{f(a)} \subseteq \Sigma_b$ and $\Sigma_{f(c)} \subseteq \Sigma_b$. Hence $\Sigma_{f(a \vee c)} = \Sigma_{f(a) \vee f(c)} = \Sigma_{f(a)} \cup \Sigma_{f(c)} \subseteq \Sigma_b$ which implies $a \vee c \in \langle f \rangle_b$.Thus $\langle f \rangle_b$ is closed under join.

Now let $a \in \langle f \rangle_b$ and let $x \leq a$ in L. By order preserving property of frame homomorphism, we have $f(x) \leq f(a)$ and hence $\Sigma_{f(x)} \subseteq \Sigma_{f(a)} \subseteq \Sigma_b$. Thus $x \in \langle f \rangle_b$. So $\langle f \rangle_b$ is lower closed and hence $\langle f \rangle_b$ is an ideal on L.

Suppose Σ_b is a meet irreducible element in $\Omega(Sp(L))$ and $a \wedge c \in \langle f \rangle_b$. Then $\Sigma_{f(a)} \cap \Sigma_{f(c)} = \Sigma_{f(a) \wedge f(c)} = \Sigma_{f(a \wedge c)} \subseteq \Sigma_b$. Since Σ_b is a meet irreducible, either $\Sigma_{f(a)} \subseteq \Sigma_b$ or $\Sigma_{f(c)} \subseteq \Sigma_b$. Hence from the definition $\langle f \rangle_b$ is a prime ideal. □

Example 3. *Consider the frame L given below.*



Here, $f : L \to L$ is defined by

$$f(x) = \begin{cases} 0 & \text{if } x=0,a \\ c & \text{if } x=b,c \\ 1 & \text{if } x=1 \end{cases}$$

Then $f$is a frame homomorphism. We have $\Sigma_0 = \phi, \Sigma_a = \{F_1\}, \Sigma_b = \{F_2\}$, $\Sigma_c = \{F_1, F_2\}, \Sigma_1 = \{F_1, F_2, F_3\}$, where completely prime filters F_1, F_2 are given by $F_1 = \{a, c, 1\}, F_2 = \{b, c, 1\}, F_3 = \{1\}$. Then $\langle f\rangle_0 = \{0, a\}$.

$$\begin{aligned} \langle f\rangle_a &= \{0, a\} \\ \langle f\rangle_b &= \{0, a\} \\ \langle f\rangle_c &= \{0, a, b, c\} \\ \langle f\rangle_1 &= L. \end{aligned}$$

Lemma 3.2.

(i) *If $b \leq c$ in L, then $\langle f\rangle_b \subseteq \langle f\rangle_c$*

(ii) *$\langle f\rangle_b = \langle f\rangle_c$ if and only if $\langle f\rangle_{b\wedge d} = \langle f\rangle_{c\wedge d}$, $\forall d \in L$.*

Proof.
(i) $b \leq c \Rightarrow \Sigma_b \subseteq \Sigma_c$.

$$\begin{aligned} a \in \langle f\rangle_b &\Rightarrow \Sigma_{f(a)} \subseteq \Sigma_b \\ &\Rightarrow \Sigma_{f(a)} \subseteq \Sigma_c \\ &\Rightarrow a \in \langle f\rangle_c \end{aligned}$$

(ii) Let $\langle f\rangle_b = \langle f\rangle_c$ Now,

$$\begin{aligned} x \in \langle f\rangle_{b\wedge d} &\Leftrightarrow \Sigma_{f(x)} \subseteq \Sigma_{b\wedge d} \\ &\Leftrightarrow \Sigma_{f(x)} \subseteq \Sigma_b \text{ and } \Sigma_{f(x)} \subseteq \Sigma_d \\ &\Leftrightarrow x \in \langle f\rangle_b = \langle f\rangle_c \text{ and } \Sigma_{f(x)} \subseteq \Sigma_d \\ &\Leftrightarrow \Sigma_{f(x)} \subseteq \Sigma_{c\wedge d} \\ &\Leftrightarrow x \in \langle f\rangle_{c\wedge d} \end{aligned}$$

Thus $\langle f\rangle_{b\wedge d} = \langle f\rangle_{c\wedge d}$. Now let $\langle f\rangle_{b\wedge d} = \langle f\rangle_{c\wedge d} \quad \forall d \in L$. Then $\langle f\rangle_{b\wedge 1} = \langle f\rangle_{c\wedge 1}$. Thus $\langle f\rangle_b = \langle f\rangle_c$. ☐

Lemma 3.3. *If f and g are two frame homomorphism on L with the property $f \leq g$, then $(g)_b \subseteq \langle f\rangle_b$*

Proof. $x \in (g)_b \Rightarrow \Sigma_{g(x)} \subseteq \Sigma_b \Rightarrow \Sigma_{f(x)} \subseteq \Sigma_{g(x)} \subseteq \Sigma_b \Rightarrow x \in \langle f\rangle_b$
Hence $(g)_b \subseteq \langle f\rangle_b$. ☐

4. Locale from the ideals $\langle f\rangle_b$

Theorem 4.1. *The set $J_f = \{\langle f\rangle_b : b \in L\}$ gives a locale.*

Proof. Define $\preceq$ on J_f by $\langle f\rangle_b \preceq \langle f\rangle_c$ if $b \leq c$ in L or $\langle f\rangle_b = \langle f\rangle_d$ for some $d \leq c$. We can easily verify that the relation $\preceq$ is a partial order on J_f. Then, $\langle f\rangle_{a\wedge b} \preceq \langle f\rangle_a$ and $\langle f\rangle_{a\wedge b} \preceq \langle f\rangle_b$ follows from the definition.Thus $\langle f\rangle_{a\wedge b}$ stands as a lower bound to both of $\langle f\rangle_a$ and $\langle f\rangle_b$.

For any other lower bound$\langle f\rangle_d$, we have $\langle f\rangle_a$ and $\langle f\rangle_b$. That is let $\langle f\rangle_d \preceq \langle f\rangle_a$ and $\langle f\rangle_d \preceq \langle f\rangle_b$. If $d \leq a$ and $d \leq b$, then we have $\langle f\rangle_d \preceq \langle f\rangle_{a\wedge b}$.

Now let $d \leq a$ and $\langle f\rangle_d = \langle f\rangle_c$ for some $c \leq b$. Then, $x \in \langle f\rangle_d \Rightarrow x \in \langle f\rangle_c \Rightarrow \Sigma_{f^*(x)} \subseteq \Sigma_c$. Since $d \leq a$, $x \in \langle f\rangle_d \Rightarrow \Sigma_{f(x)} \subseteq \Sigma_d \subseteq \Sigma_a$. Thus $\langle f\rangle_d \subseteq \langle f\rangle_{c\wedge a}$.

Since $c \wedge a \leq c$, $\langle f\rangle_{c\wedge a} \subseteq \langle f\rangle_c = \langle f\rangle_d$.

Thus $\langle f\rangle_{c\wedge a} = \langle f\rangle_d$ where $c \wedge a \leq a \wedge b$. Hence $\langle f\rangle_d \preceq \langle f\rangle_{a\wedge b}$. This is true for other two cases. Hence $\langle f\rangle_a \wedge \langle f\rangle_b = \langle f\rangle_{a\wedge b}$.

In a similar manner,$\bigvee\langle f\rangle_{a_i} = \langle f\rangle_{\bigvee a_i}$.

Now, $\langle f\rangle_b \wedge \bigvee\langle f\rangle_{a_i} = \langle f\rangle_{b\wedge\bigvee a_i} = \langle f\rangle_{\bigvee(b\wedge a_i)} = \bigvee\langle f\rangle_{b\wedge a_i} = \bigvee(\langle f\rangle_b \wedge \langle f\rangle_{a_i})$. Thus J_f is a locale with bottom $\langle f\rangle_0$ and top $\langle f\rangle_1$. ☐

Example 4. *The locale J_f corresponding to the frame homomorphism f in example 3 is given by $J_f = \{\langle f\rangle_0, \langle f\rangle_c, \langle f\rangle_1\}$.*

Proposition 4.1. *J_f is compact locale if and only if $\Omega(Sp(L))$ is a compact locale.*

Proof. First suppose J_f is compact locale. Let $\Sigma_1 = \Sigma_{f(1)} \subseteq \bigcup \Sigma_{b_i}$. That is $\Sigma_{f(1)} \subseteq \Sigma_{\bigvee b_i}$ which implies $1 \in \langle f \rangle_{\vee b_i}$. Since $\langle f \rangle_{\vee b_i}$ is an ideal, we have $\langle f \rangle_{\vee b_i} = \langle f \rangle_1$. Then by compactness of J_f we have $\langle f \rangle_{b_1 \vee b_2 \vee \ldots . b_n} = \langle f \rangle_1$. Then since $1 \in \langle f \rangle_{b_1 \vee b_2 \vee \ldots . b_n}$, we have $\Sigma_1 = \Sigma_{f(1)} \subseteq \Sigma_{b_1} \bigcup \Sigma_{b_2} \bigcup \ldots .. \bigcup \Sigma_{b_n}$. This shows that $\Omega(Sp(L))$ is a compact element of Sp(L).

Conversely assume $\Omega(Sp(L))$ is a compact locale. Suppose $\bigvee \langle f \rangle_{b_i} = \langle f \rangle_1$. Then clearly $1 \in \langle f \rangle_{\vee b_i}$. Thus $\Sigma_1 = \Sigma_{f(1)} \subseteq \Sigma_{\vee b_i}$. Since $\Omega(Sp(L))$ is a compact locale, we have $\Sigma_1 = \Sigma_{f(1)} \subseteq \Sigma_{b_1 \vee b_2 \vee \ldots .. \vee b_n}$. This gives $1 \in \langle f \rangle_{b_1 \vee b_2 \vee \ldots .. \vee b_n}$. Hence $\langle f \rangle_{b_1 \vee b_2 \vee \ldots .. \vee b_n} = \langle f \rangle_1$. Thus J_f is compact. ☐

Proposition 4.2. *If L is a Boolean locale, then J_f is a Boolean locale.*

5. Congruence in L with respect to the frame homomorphism f

Definition 5.1. *Let $f : L \to L$ be a a morphism in the category **Frm**. For $a, b \in L$, define a relation R_f on L as $aR_f b$ if $\langle f \rangle_a = \langle f \rangle_b$.*

Proposition 5.1. *R_f is a congruence on L.*

Proof. Clearly R_f is an equivalence relation. Also, if $(a, b) \in R_f$, then by Lemma 3.2 (ii), $(a \wedge c, b \wedge c) \in R_f$.

$$\begin{aligned}(a, b) \in R_f &\Rightarrow \langle f \rangle_a = \langle f \rangle_b \\ &\Rightarrow \langle f \rangle_a \bigvee \langle f \rangle_s = \langle f \rangle_b \bigvee \langle f \rangle_s \\ &\Rightarrow \langle f \rangle_{a \bigvee S} = \langle f \rangle_{b \bigvee S} .\end{aligned}$$

Thus $(a, b) \in R_f \Rightarrow (a \wedge c, b \wedge c) \in R_f$ and $(a \bigvee S, b \bigvee S) \in R_f$. Hence R_f is a congruence on L. ☐

Remark 5.1. *Since R_f is a congruence on L, by [4], L/R_f is a frame under the induced partial order $[x] \leq [y]$ if and only if $x \leq y$ in L.*

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