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ON SIGNED (NON-NEGATIVE) MAJORITY TOTAL DOMINATION OF SOME GRAPHS

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ABSTRACT. For a simple graph $G = (V;E)$, a two valued function $h : V \to \{-1,1\}$ is called a (non-negative) majority total dominating function if the sum of its function values over at least half the open neighbourhoods is at least (zero) one. A (non-negative) majority total domination number of a graph G is the minimum value of $\sum_{v\in V(G)} h(v)$ over all (non-negative) majority total dominating functions f of G and it is denoted by $(\gamma^{nt}_{maj}(G))\gamma^{t}_{maj}(G)$. In this paper, we have obtained exact value of majority total domination number and non-negative majority total domination number of some prism related graphs.

1. INTRODUCTION

All graphs handled in this paper is finite and simple. All graphs considered here are simple, finite and undirected graphs. For basic definition and notation we follow [1,2].

The study of domination is one of the well studied areas within graph theory. A subset L of vertices is said to be a dominating set of G if every vertex in V either belongs to L or is adjacent to a vertex in L. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. An excellent survey

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of advanced topics on domination parameters are given in the book edited by Haynes et al [2].

For a real valued function $h : V \to R$ on V, weight of h is defined to be

$$w(h) = \sum_{v \in V} h(v).$$

Further, for a subset S of V we let

$$h(S) = \sum_{v \in S} h(v).$$

Therefore $w(h) = h(V)$. A two valued function $h : V \to \{-1, 1\}$ is called a signed majority dominating function if the sum of its function values over at least half the closed neighbourhoods is at least one. A non-negative majority total domination number of a graph G is the minimum value of $\sum_{v \in V(G)} f(v)$ over all non-negative majority total dominating functions f of G and it is denoted by $\gamma^{nt}_{maj}(G)$.

Broere et al. introduced Majority domination in [3] and this concept is further studied in [4]. Later, Hua-ming xing et al. [5] introduced and studied the following concept. A function $h : V \to \{-1, 1\}$ is called a signed majority total dominating function if $h(N(v)) \geq 1$ for at least half of the vertices in graph G. The signed majority total domination number of G, is denoted by $\gamma^{t}_{maj}(G)$, and is defined as $\gamma^{t}_{maj}(G) = \{w(h) | h$ is a signed majority total dominating function of $G\}$. Further it has been studied in [6].

In 2017, Sahul Hamid and S. Anandha Prabhavathy [6] introduced non-negative majority total domination of a graph G which is defined as follows: a two valued function $h : V \to \{-1, 1\}$ is called a non-negative majority total dominating function if the sum of its function values over at least half the open neighbourhoods is at least zero. A non-negative majority total domination number of a graph G is the minimum value of $\sum_{v \in V(G)} h(v)$ over all non-negative majority total dominating functions f of G and it is denoted by $\gamma^{nt}_{maj}(G)$, see Figure 1. So for exact values of $\gamma^{nt}_{maj}(G)$ are known only for complete graph, complete bipartite graph, path, cycle and star.

Before we get into results, we define some notation which we have used throughout the paper. If h is a minimum (non-negative) majority total dominating function, then

- $V_+ = \{v \in (G) : h(v) = +1\}$;

- $V_- = \{v \in V(G) : h(v) = -1\}$; and
- $N_h = \{v \in V(G) : h(h(v)) \geq 1\}(\{v \in V(G) : N(h(v)) \geq 0\})$.

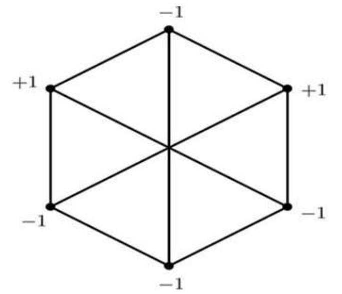

Figure 1: The graph G with $\gamma^{nt}_{maj}(G) = -2$

The prism over a graph G of x vertices is a Cartesian product of $G\Box K_2$.

- Let $V(G\Box K_2) = \{a_1, a_2, ..., a_x\} \cup \{b_1, b_2, ..., b_x\}$ and
- $E(G\Box K_2) = \{a_ia_j$ and $b_ib_j|v_iv_j \in E(G)\} \cup \{a_ib_i|1 \leq i \leq x\}$.

2. Majority total domination number of prism graph

In [6], exact value of $\gamma^t_{maj}(K_x\Box K_2)$ is obtained for odd integer x. In the following theorem we have completed for even values of x.

Theorem 2.1. *Let $x \geq 4$ be an even integer. Then $\gamma^t_{maj}(K_x\Box K_2) = 4 - x$.*

Proof. Let $A = \{a_1, a_2, ..., a_x\}$ and $B = \{b_1, b_2, ..., b_x\}$ be the sets of vertices of the 2-copies of K_x in $K_x\Box K_2$. For $k \geq 2$, let $x = 2k$. Consider a function $h : A \cup B \to \{-1, 1\}$ by

$$h(a_i) = \begin{cases} +1 & \text{if} \quad 1 \leq i \leq k+2 \\ -1 & \text{otherwise} \end{cases}, \tag{2.1}$$

and $h(b_j) = -1$ for all j. It is easy to verify that $h(N(v)) \geq 1$ for at least half of the vertices with weight $w(h) = k + 2 - (k - 2 + 2k) = k + 4 - 3k = 4 - x$. Hence

$$\gamma^t_{maj}(K_x\Box K_2) \leq 4 - x. \tag{2.2}$$

To prove the reverse inequality, let g be a non-negative majority total dominating function of $K_x\Box K_2$.

Claim: $N_g \cap A = \phi$ or $N_g \cap B = \phi$. Suppose $N_g \cap A = \phi$ or $N_g \cap B = \phi$. Then there exist vertices a_i and b_j belongs to N_h such that $g(N(a_i)) \geq 0$ and $g(N(b_j)) \geq 0$. This implies $g(a_i) = 1$ for at least $\frac{x}{2}$ vertices of A and $g(b_j) = 1$

for at least $\frac{x}{2}$ vertices of B. Thus $|V_+| \geq \frac{x}{2} + \frac{x}{2} = x$. By equation (2.1), we have $|V_+| = k + 2 < x$. Consequently, $h(V) < g(v) = \gamma^t_{maj}(K_x \overline{\square} K_2)$ which is a contradiction. Hence $N_h \cap A = \phi$ or $N_h \cap B = \phi$. By above claim we have $N_g \cap A = \phi$ or $N_g \cap B = \phi$. W.L.O.G we assume that $N_g \cap B = \phi$. Since g is a minimum majority total domination function of $K_x \overline{\square} K_2$ and $|N_g| \geq x, N_g = A$. Since $g(V) = \gamma^t_{maj}(K_x \overline{\square} K_2)$, there exists a vertex b_j belongs to B such that $g(b_j) = -1$. Clearly, $a_j \in N_g$ and hence $|V_+ \cap (A - \{a_j\})| \geq k + 1$. Thus $|V_+| \geq k + 2$. Hence we have

$$\gamma^t_{maj}(K_x \overline{\square} K_2) \leq 4 - x. \tag{2.3}$$

From (2.2) and (2.3), we have $\gamma^t_{maj}(K_x \overline{\square} K_2) = 4 - x$. □

Theorem 2.2. *For integers $x \geq 3$,*

$$\gamma^t_{maj}(K_{x,x} \overline{\square} K_2) = \begin{cases} 4 - 2x & \textit{if} \quad x \quad \textit{is even} \\ 6 - 2x & \textit{if} \quad x \quad \textit{is odd} \end{cases}.$$

Proof. Let $A = \{a_1, a_2, ..., a_x\}$ and $B = \{b_1, b_2, ..., b_x\}$ be the bipartition of one copy of $K_{x,x}$ of $K_{x,x} \overline{\square} K_2$ and let $C = \{c_1, c_2, ..., c_x\}$ and $D = \{d_1, d_2, ..., d_x\}$ be the bipartition of another copy of $K_{x,x}$ of $K_{x,x} \overline{\square} K_2$.

Case 1: x is even. Let $x = 2k$. Define a function $h : V(K_{x,x} \overline{\square} K_2) \to \{-1, +1\}$ as follows:

$$h(a_i) = h(c_i) = \begin{cases} 1 & \text{if} \quad 1 \leq i \leq k + 1 \\ -1 & \text{otherwise} \end{cases}$$

and for $1 \leq j \leq x, h(b_j) = h(d_j) = -1$. Clearly, $h(N(v)) \geq 1$ for at least half of the vertices with weight $w(h) = 2(k + 1) - 2x - 2(k - 1) = 4 - 2x$. Hence

$$\gamma^t_{maj}(K_{x,x} \overline{\square} K_2) \leq 4 - x. \tag{2.4}$$

Case 2: x is odd. Let $x = 2k + 1$. Define a function $h : V(K_{x,x} \overline{\square} K_2) \to \{-1, +1\}$ as follows:

$$h(a_i) = h(c_i) = \begin{cases} 1 & \text{if} \quad 1 \leq i \leq k + 2 \\ -1 & \text{otherwise} \end{cases}$$

and for $1 \le j \le x, h(b_j) = h(d_j) = -1$. Clearly, $h(N(v)) \ge 1$ for at least half of the vertices with weight $w(h) = 2(k+2) - 2x - 2(k-1) = 6 - 2x$. Hence

$$\gamma^t_{maj}(K_{x,x}\overline{\Box}K_2) \le 6 - x. \tag{2.5}$$

On the other hand, let h be any minimum majority total domination function of $K_{x,x}\overline{\Box}K_2$. It is clear that $|V_+| - |V_-| \ge 1$ and $|V_+| + |V_-| = x + 1$ for each vertex $v \in N_h$. Consequently, $|V_+| \ge \lceil \frac{n+2}{2} \rceil$. Thus

$$\gamma^t_{maj}(K_{x,x}\overline{\Box}K_2) \ge \begin{cases} 4 - 2x & \text{if} \quad x \quad \text{is even} \\ 6 - 2x & \text{if} \quad x \quad \text{is odd} \end{cases}. \tag{2.6}$$

From (2.4), (2.5) and (2.6), we have

$$\gamma^t_{maj}(K_{x,x}\overline{\Box}K_2) = \begin{cases} 4 - 2x & \text{if} \quad x \quad \text{is even} \\ 6 - 2x & \text{if} \quad x \quad \text{is odd.} \end{cases}$$

□

3. Non-negative majority total domination number of prism graph

In this section we have obtained exact value of γ^{nt}_{maj} of prism over K_x.

Theorem 3.1. *Let $x \ge 3$ be an odd integer. Then $\gamma^{nt}_{maj}(K_x\overline{\Box}K_2) = 3 - x$.*

Proof. Let $A = \{a_1, a_2, ..., a_x\}$ and $B = \{b_1, b_2, ..., b_x\}$ be the sets of vertices of the 2-copies of K_x in $K_x\overline{\Box}K_2$. For $k \ge 2$, let $x = 2k + 1$. Define a function $h : A \cup B \to \{-1, 1\}$ by

$$h(a_i) = \begin{cases} +1 & \text{if} \quad 1 \le i \le k+2 \\ -1 & \text{otherwise} \end{cases} \tag{3.1}$$

and $h(b_j) = -1$ for all j. It is not difficult to check that $h(N(v)) \ge 0$ for at least half of the vertices with weight $w(h) = k+2-(k-1+2k+1) = k+2-3k = 3-x$. Hence

$$\gamma^{nt}_{maj}(K_x\overline{\Box}K_2) \le 3 - x. \tag{3.2}$$

On the other hand, let g be a non-negative majority total dominating function of $K_x\overline{\Box}K_2$. To prove $\gamma^{nt}_{maj}(K_x\overline{\Box}K_2) \ge 3 - x$, we need to prove the following claim.

Claim: $N_g \cap A = \phi$ or $N_g \cap B = \phi$. Suppose $N_g \cap A = \phi$ or $N_g \cap B = \phi$. Then there exist vertices a_i and b_j belongs to N_h such that $g(N(a_i)) \geq 0$ and $g(N(b_j)) \geq 0$. This implies $g(a_i) = 1$ for at least $\left\lfloor \frac{x}{2} \right\rfloor$ vertices of A and $g(b_j) = 1$ for at least $\left\lfloor \frac{x}{2} \right\rfloor$ vertices of B. Thus $|V_+| \geq \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{2} \right\rfloor = 2\left\lfloor \frac{x}{2} \right\rfloor$. By equation (3.1), we have $|V_+| = k+2 < 2\left\lfloor \frac{x}{2} \right\rfloor$. Consequently, $h(V) < g(v) = \gamma^{nt}_{maj}(K_x \Box K_2)$ which is a contradiction. Hence either $N_h \cap A = \phi$ or $N_h \cap B = \phi$. By above claim we have $N_g \cap A = \phi$ or $N_g \cap B = \phi$. W.L.O.G we assume that $N_g \cap B = \phi$. Since g is a NMTDF of $K_x \Box K_2$ and $|N_g| \geq x$, $N_g = A$. It is easy to observe that $|V_+| \geq k+2$. Hence we have

$$\gamma^{nt}_{maj}(K_x \Box K_2) \leq 3 - x. \tag{3.3}$$

From (3.2) and (3.3), we have $\gamma^{nt}_{maj}(K_x \Box K_2) = 3 - x$. □

Theorem 3.2. *Let $x \geq 4$ be an even integer. Then $\gamma^{nt}_{maj}(K_x \Box K_2) = 2 - x$.*

Proof. Let $A = \{a_1, a_2, ..., a_x\}$ and $B = \{b_1, b_2, ..., b_x\}$ be the sets of vertices of the 2-copies of K_x in $K_x \Box K_2$. For $k \geq 2$, let $x = 2k$. Define a function $h : A \cup B \to \{-1, 1\}$ by

$$h(a_i) = \begin{cases} +1 & \text{if} \quad 1 \leq i \leq k+1 \\ -1 & \text{otherwise} \end{cases}$$

and $h(b_j) = -1$ for all j. It is not difficult to check that $h(N(v)) \geq 0$ for at least half of the vertices with weight $w(h) = k+1-(k-1+2k) = k+2-3k = 2-x$. Hence

$$\gamma^{nt}_{maj}(K_x \Box K_2) \leq 2 - x. \tag{3.4}$$

On the other hand, let g be a non-negative majority total dominating function of $K_x \Box K_2$. It is easy to observe that $N_g \cap A = \phi$ or $N_g \cap B = \phi$ (for detail proof see the proof of the claim in Theorem 3.1). W.L.O.G we assume that $N_g \cap B = \phi$. Since g is a NMTDF of $K_x \Box K_2$ and $|N_g| \geq x, N_g = A$. It is easy to observe that $|V_+| \geq k+1$. Hence we have

$$\gamma^{nt}_{maj}(K_x \Box K_2) \leq 2 - x. \tag{3.5}$$

From (3.4) and (3.5), we have $\gamma^{nt}_{maj}(K_x \Box K_2) = 2 - x$. □

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