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GRAPHS ASSOCIATED WITH QUANTALES

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ABSTRACT. In this paper we study the graphs associated with Quantales. Quantales are the lattice theoretic counterpart of semi rings. In literature, a wide variety of papers are available associating graphs with rings. We extend this idea to quantales.A quantale is a complete sup-lattice Q together with an associative binary operation $*$ satisfying the infinite distributive laws i) $a*(\vee_\alpha b_\alpha) = \vee_\alpha(a*b_\alpha)$ and ii) $(\vee_\alpha b_\alpha)*a = \vee_\alpha(b_\alpha * a)$ for all $a \in Q$ and $\{b_\alpha\} \subseteq Q$. We define a map called deductions on Quantales and develop ideals of the form $[a]^d = \{x \in Q : d(x*a) = 0\}$ with respect to the deduction on the quantale. We study the properties of graph associated with these ideals. We define an equivalence relation on the collection of ideals $[a]^d$ and we observe that the associated graph G(Q,d) is a disconnected graph. We have also constructed the zero divisor graph Z_G and have studied some of its properties.

1. INTRODUCTION

Quantales were introduced by C. J. Mulvey to provide a lattice theoretic setting for studying C^* algebras.This can also be viewed as the quantisation of the term locale. The concept of quantale was introduced to study the ideal theory of rings in terms of a lattice of ideals with a multiplication. Algebraically, quantales can be viewed as semirings,which enables us to introduce graph theoretical ideas to the theory of quantales. This paper deals with the graphs which are developed through maps called deductions.

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In the first section, we deal with the basic definitions in quantale theory and in graph theory needed for the development of the paper.Also, we define the map deductions and the $*$ ideal $[a]^d = \{x \in Q : d(x * a) = 0\}$. The second section introduces graph theory into the realm of quantales through the graph $G(Q, d)$. The third sections deals with the zerodivisor graph Z_G and some of its properties.

2. PRELIMINARIES

Definition 2.1. *[6] A frame is a complete lattice L satisfying the infinite distributivity law $a \sqcap \bigsqcup B = \bigsqcup\{a \sqcap b; b \in B\}$ for all $a \in L$ and $B \subseteq L$.*

Example 1. *The lattice of open sets of a topological space forms a frame*

Definition 2.2. *[4] A quantale is a complete lattice Q with an associative binary operation $*$ satisfying i) $a * (\vee_\alpha b_\alpha) = \vee_\alpha(a * b_\alpha)$ and ii) $(\vee_\alpha b_\alpha) * a = \vee_\alpha(b_\alpha * a)$ for all $a \in Q$ and $\{b_\alpha\} \subseteq Q$.*

Definition 2.3. *[4] A quantale Q is commutative if and only if $a * b = b * a$ for every $a, b \in Q$.*

Example 2. *[4]*

(i) *Any frame is a quantale with $* = \wedge$. It is commutative, idempotent, unital with unit T.*
(ii) *Sub(R),the set of additive subgroups of R is a quantale with $sup = \Sigma$ and with $A * B = AB = \{a_1b_1 + a_2b_2 + ...a_nb_n : a_i \in A, b_i \in B\}$.*

Definition 2.4. *[2] Let I be a subset of a quantale Q. I is called a left (respectively right) ideal of Q if*

(i) *$X \subseteq I$ implies $\bigvee X \in I$,*
(ii) *$x \in I$ and $y \leq x$ then $y \in I$,*
(iii) *$x \in I$ imples $a * x \in I$ (resp $x * a \in I$) for all $a \in Q$.*

Definition 2.5. *Let I be a subset of a quantale Q. I is called a left (right) $*$ ideal of Q if*

(i) *$X \subseteq I$ implies $\bigvee X \in I$,*
(ii) *$x \in I$ imples $a * x \in I$ (resp $x * a \in I$) for all $a \in Q$.*

The idea of connecting graph theory and algebraic strucutres started with the work of I.Beck.Here we introduce some definitions and theorems important for our study.

Definition 2.6. *[1] Let $G = (V, E)$ be a graph with vertex set V and edge set E. A vertex colouring of G with vertex set V is a map $f : V \to S$, where S is a set of distinct colours;it is proper if adjacent vertices of G recieve distinct colours of S; that is, $uv \in E(G)$, then $f(u) \neq f(v)$.*

Definition 2.7. *[1] The chromatic number of a graph G, denoted by $\chi(G)$, is the minimum number of colours needed for a proper vertex colouring of G. G is k-chromatic ,if $\chi(G) = k$.*

Definition 2.8. *[1] A graph is bipartite if its vertex set can be partitioned into two nonempty subsets X and Y such that each edge of G has one end in X and the other in Y.The pair (X, Y) is called a bipartition of the bipartite graph. The bipartite graph with bipartition (X, Y) is denoted by $G(X, Y)$. A simple bipartite graph is complete if each vertex of X is adjacent to all vertices of Y. If $G(X, Y)$ is complete with $\mid X \mid = p$ and $\mid Y \mid = q$, then $G(X, Y)$ is denoted by $K_{p,q}$. A complete bipartite graph of the form $K_{1,q}$ is called a star.*

Definition 2.9. *[1] A clique of G is a complete subgraph of G. A clique of G is a maximal clique of G if it is not properly contained in another clique of G. The order of a maximum clique of G is called the clique number of G and is denoted by $\omega(G)$.*

Theorem 2.1. *[1] A graph is bipartite if and only if it contains no odd cycles.*

Theorem 2.2. *[3] Let G be a non empty graph . Then $\chi(G) = 2$ if and only if G is bipartite.*

Theorem 2.3. *[3] Let G be a graph .Then $\chi(G) \geq 3$ if and only if G has an odd cycle.*

We define and discuss some properties of the map called deductions. The map deductions and its detailed study has been done in [5].

Definition 2.10. *[5] Consider the quantale $(Q, \bigvee, \wedge, *)$. A map $d : Q \to Q$ on a quantale Q is called a deduction on Q if it satisfies the following conditions.*

(i) $d(\bigvee_{i\in I} b_i) = \bigvee_{i\in I} d(b_i)$, *where I is some indexed set.*

(ii) $d(a * b) = a * d(b)$ *, this property is called translation in second variable.*

If Q is commutative we impose the additional condition that $d(a * b) = a * d(b) = d(a) * b$.

Lemma 2.1. *[5] Let d be a deduction on a quantale $(Q, \vee, \wedge, *)$. For any $x, y \in Q$ we have*

(i) $d(0) = 0$,
(ii) $x \leq y$ *implies* $d(x) \leq d(y)$.

Definition 2.11. *[5] Let d be a deduction on Q then define $[a]^d = \{x \in Q : d(x * a) = 0\}$. Or , equivalently, $[a]^d = \{x \in Q : x * d(a) = 0\}$.*

Lemma 2.2. *[5] $[a]^d$is a left sided $*$-ideal of Q.*

Theorem 2.4. *[5] If Q is commutative then $[a]^d$ is a $*$-ideal of Q.*

Theorem 2.5. *[5] Let d be a deduction on Q, then for any $a, b \in Q$, we have the following*

(i) $a \leq b \Rightarrow [b]^d \subseteq [a]^d$,
(ii) $[a \vee b]^d = [a]^d \cap [b]^d$.

Now we introduce the graphs on Quantales.

3. Graphs on Quantales with respect to Deductions

In this section we introduce the theory of graphs into Quantales. We are familiar with the interplay of ring theory and graphs. Similarly, we investigate the possible development of graphs in the theory of Quantales.

The Graph $G(Q, d)$

Definition 3.1. *Let Q be a quantale. A graph $G = (V_G, E_G)$ where the vertex set $V_G = Q$ and the edge set $E_G = \{(a, b) : [a]^d = [b]^d, a \neq b\}$ is called the âĂŸgraph with respect to the deduction dâĂŹ and is denoted as $G(Q, d)$.*

Theorem 3.1. *$G(Q, d)$ is always a disconnected graph.*

Proof. Let us define a relation θ_d on Q such that $(a, b) \in \theta_d$ if and only if $[a]^d = [b]^d$. It can be easily shown that θ_d is an equivalence relation on Q. If

$(a,b) \in E_G$ then $(a,b) \in \theta_d$ and thus belongs to the same equivalence class. Every equivalence class is either equal or disjoint. Hence, $G(Q,d)$ is always a disconnected graph. □

Let $\mathbb{E}(Q,\theta_d)$ denote the set of all equivalence classes with the respect to θ_d and $\mathbb{C}(G)$ denote the collection of components of $G(Q,d)$.

Theorem 3.2. *$\mathbb{E}(Q,\theta_d)$ and $\mathbb{C}(G)$ are equivalent sets.*

Proof. Let $X \in \mathbb{E}(Q,\theta_d)$. Since X is nonempty, there exists some $a \in X$. The vertex a is in one of the connected components of $G(Q,d)$, say $C(a)$. All other vertices in $C(a)$ will fall in the same equivalence class X. We define $f : \mathbb{E}(Q,\theta_d) \to \mathbb{C}(G)$ as $f(X) = C(a)$. Clearly, f is well defined. Let X, $Y \in \mathbb{E}(Q,\theta_d)$ such that $f(X) = f(Y)$. Let $a \in X$ and $b \in Y$, then $C(a) = f(X) = f(Y) = C(b)$. That is, a and b belong to the same connected component, and so $[a]^d = [b]^d$ implying that they lie in the same equivalence class. Hence $X = Y$. To prove f is onto, consider any connected component, say D. Let p be any vertex in D. Then $D = C(p)$. Let $Z \in \mathbb{E}(Q,\theta_d)$ be such that $p \in Z$. Hence $f(Z) = D$. Therefore $\mathbb{E}(Q,\theta_d)$ and $\mathbb{C}(G)$ are equivalent sets. □

Corollary 3.1. *For any quantale Q and a deduction d on Q, $\omega(G(Q,d)) = |\mathbb{E}(Q,\theta_d)|$.*

Theorem 3.3. *For the graph $G(Q,d)$, we observe the following properties:*

(1) *Each equivalence class $X \in \mathbb{E}(Q,\theta_d)$ will become a connected component.*
(2) *Each component is a complete subgraph of $G(Q,d)$.*
(3) *The number of vertices in each component is equal to $|X|$.*

Proof. 1) and 3) follow easily from the previous theorem, so we prove only the second property. Every $X \in \mathbb{E}(Q,\theta_d)$ is mapped to a component $f(X)$ of $G(Q,d)$. Let $X = \{u_1, u_2, u_3, \cdots, u_k\}$, then $[u_i]^d = [u_j]^d, \forall i,j \in \{1,2,\cdots,k\}$. Therefore $(u_i,u_j) \in E_G$, $\forall i,j \in \{1,2,k\}$. Thus the component $f(X)$ is complete. □

4. The Graph Z_G

In this section, we introduce a different type of graph using the map deduction on a commutative quantale Q. We observe that the graph developed is always connected, in particular it can be a star graph.

Definition 4.1. *An elemmaent a of a commutative quantale Q is said to be a zero divisor if there exist $b \neq 0$ such that $a * b = 0$.*

Definition 4.2. *If the deduction d is the identity map on a commutative quantale Q then $[a]^d = \{x \in Q | d(x * a) = 0\}$ = $\{x \in Q | x * a = 0 = a * x\}$. We introduce two new terminologies the zero set of a and the closure set of a.*
The zero set of a is defined and denoted as $[a]^\circ = \{x \in Q | x \neq 0, x * a = 0\}$.
The closure set of a is defined and denoted as $(\overline{[a]}) = \{a\} \cup [a]^\circ$.

Remark 4.1.
1. If $[a]^\circ \neq \emptyset$, then both the sets $[a]^\circ$ and $\overline{[a]}$ are lower sets of Q.
2. If $[a]^\circ \neq \emptyset$ then $[a]^\circ$ is an ideal of Q.

Definition 4.3. *Let Q be a commutative quantale. A graph $G = (V, E)$ where the vextex set $V = Q$ and the edge set $E = \{(a, b) : \overline{[a]} \cap \overline{[b]} \neq \emptyset, a \neq b\}$ is called the âĂŸZero-graph of QâĂŹ and is denoted as $\mathbf{Z_G}$.*

Lemma 4.1. *For a commutative quantale Q, $\mathbf{Z_G}$ is always connected.*

Proof. Since $0 * a = 0, \forall a \in Q$, we have $[0]^\circ = Q$. Hence 0 has an edge with every other vertices of $\mathbf{Z_G}$. □

Theorem 4.1. *For a commutative quantale Q, $\mathbf{Z_G}$ is either a star graph or contains a cycle C_3.*

Proof. Since, 0 has an edge with every other vertices of $\mathbf{Z_G}$, it will always contain a star graph with internal node at 0. If there is any other edge $(a, b) \in E$ in the graph then $0 - a - b - 0$ will form the triangle C_3 □

Corollary 4.1. *If Q has zero divisors then $\mathbf{Z_G}$ is not bipartite.*

The following theorem is obtained from the above observations.

Theorem 4.2. *If Q does not have any nonzero divisors then $\mathbf{Z_G}$ will have the following properties:*

(1) *$\mathbf{Z_G}$ is a star graph.*
(2) *$\mathbf{Z_G}$ is a bipartite graph.*
(3) *The chromatic number $\chi(\mathbf{Z_G})$=2.*

Theorem 4.3. *Let $y \in Q$ be such that $[y]^\circ \neq \emptyset$, then the subgraph induced by the set $\downarrow y \subseteq V$, is a clique of $\mathbf{Z_G}$.*

Proof. Let $x \in [y]^{\circ}$. For $a, b \in \ \downarrow y$, $a * x \leq y * x$, implies $a * x = 0$. Similarly $b * x \leq y * x$ implies $b * x = 0$. Hence $x \in [a]^{\circ} \cap [b]^{\circ}$ and $\overline{([a])} \cap \overline{([b])} \neq \emptyset$. Therefore any two vertices of $\downarrow y \subseteq V$ is always connected by an edge. Thus the subgraph induced by the set $\downarrow y \subseteq V$ is a clique of $\mathbf{Z_G}$. □

Theorem 4.4. *If $\mathbf{Z_G}$ is not a star graph then $\chi(\mathbf{Z_G}) \geq 3$.*

Proof. If $\mathbf{Z_G}$ is not a star graph then it will contain an odd cycle. Hence the result follows. □

5. Conclusion

We have made an attempt to detail a graph theoretic approach to quantale theory. This research paper is just a beginning in this direction .We have exploited the close relation between quantales and semirings. We can study the usual notions of zero divisor graphs, total graphs and so on for quantales. Quantales can be infinite , the applications of graphs associated with the quantales can be found in networking problems. Star graphs finds application in computer networks. Also, it is used as local model of curves in tropical geometry. The theory of quantales may be extended into such broader areas.

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