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ON EXTREME ENTRIES OF PRINCIPAL EIGENVECTOR OF A GRAPH

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ABSTRACT. Let $G=(V,E)$ be a simple, connected graph. In this paper, we present some lower bound for the maximal entry of the principal eigenvector corresponding to the spectral radius λ_1 of G. Also, we find some lower bound for the ratio of the maximal entry to the minimal entry of the principal eigenvector. Moreover, we present some examples where our lower bounds are better than the bounds given by Cioaba and Gregory [3], Nikiforov [7] and Zhang [11].

1. INTRODUCTION

Let $G=(V,E)$ be a simple, connected graph with p vertices and q edges. The degree d_i of a vertex $v_j \in V$ $(i,j=1,2,...,p)$ is the number of vertices adjacent to it. Denote $\Delta(G)=\max\{d_i|i=1,2...,p\}$ and $\delta(G)=\min\{d_i|i=1,2,...,p\}$. The adjacency matrix $A(G)$ of G is the square matrix of order p defined by its entries as $a_{ij}=1$ if $v_iv_j \in E$ and zero otherwise. Let $\lambda_i (i=1,2,..,p)$ be the eigenvalues of G with the order $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_p$. The largest eigenvalue λ_1 of $A(G)$ is called the spectral radius of G. If $V_1=\left(x_{1_1},x_{1_2},...,x_{1_p}\right)^T$ is a positive eigenvector of $A(G)$ with unit length corresponding to λ_1, then it is called the principal eigenvector. Applications of these vectors are extensively studied in [2], [5], [6], [9]. It is clear that if G is regular, then the spectral radius

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λ_1 coincides to the degrees $\Delta = \delta$. Also the ratio $\frac{x_i}{x_j} = 1(i \neq j, i, j = 1, 2, ..., p)$, since $x_{1_1} = x_{1_2} = ... = x_{1_p} = \frac{1}{\sqrt{p}}$. Therefore, studying bounds for such ratios in case of a non-regular graph is of ample interest. In case of social networks [9] and in the fields of numerical analysis [5], the estimates of the ratio of entries of the principal eigenvector of G is used. Various bounds for the maximal entry x_{max} and the ratio $\frac{x_{max}}{x_{min}}$ of maximal entry x_{max} to the minimal entry x_{min} of the principal eigenvector of λ_1 have been investigated in [3,7,8] and [11]. For basic definitions and results of various graphs, we refer [10].

In this paper, we estimate a new lower bound on x_{max} for a complete multipartite graph in terms of spectral radius λ_1 and δ, in the section 3. Also we present a lower bound on the ratio $\frac{x_{max}}{x_{min}}$ for a non-regular graph, in the section 4. Moreover, we compare our bounds with some known bounds from [3], [7] and [11].

2. Preliminaries

In this section, we shall enlist some results and previously known bonds for x_{max} and $\frac{x_{max}}{x_{min}}$.

Lemma 2.1. *[4] If $V_i = (x_{i_i}, x_{i_2}, ..., x_{i_p})(i = 1, 2, ..., p)$ are the orthonormal eigenvectors corresponding to $\lambda_i(i = 1, 2, ..., p)$ respectively in a multi-diagraph G and $C_i = \Big(\sum_{j=1}^{p} x_{i_j}\Big)^2$, then*

$$W_k(G) = \lambda_1^k C_1 + \lambda_2^k C_2 + ... + \lambda_p^k C_p \tag{2.1}$$

where $W_k(G)$ is the total number of k-length walks in G.

Lemma 2.2. *[4] A graph G contains exactly one positive eigenvalue if and only if G is a complete multipartite graph with no isolated vertices.*

Lemma 2.3. *[3] Let G be a non-regular graph and x_{max} be the maximal entry of the principal eigenvector corresponding to λ_1, then*

$$\frac{1}{\sqrt{p - \frac{1}{\Delta}}} < x_{max} \tag{2.2}$$

Lemma 2.4. *[3] Let G be a non-regular graph and x_{min} be the minimal entry of the principal eigenvector corresponding to λ_1, then*

$$x_n \leq \frac{(\Delta - \lambda_1)}{\sqrt{p}(\Delta - \bar{d})}, \tag{2.3}$$

where $\bar{d} = \dfrac{\sum_{i=1}^{p} d_i}{p}$.

Lemma 2.5. *[1] Let* $a_1, a_2, ..., a_p$ *and* $b_1, b_2, ..., b_p$ *be real numbers such that* $0 < a \leq a_i \leq A$ *and* $0 < b \leq b_i \leq B, \forall i = 1, 2, ..., p$*, then*

$$|p\sum_{i=1}^{p} a_i b_i - \sum_{i=1}^{p} a_i \sum_{i=1}^{p} b_i| \leq \alpha(p)(A-a)(B-b)\,, \tag{2.4}$$

where $\alpha(p) = p\left[\frac{p}{2}\right]\left(1 - \frac{1}{p}\left[\frac{p}{2}\right]\right)$ *and the equality in* (2.4) *holds if and only if* $a_1 = a_2 = ... = a_p$ *and* $b_1 = b_2 = ... = b_p$.

Lemma 2.6. *[11] Let* G *be a non-regular graph and* x_{max}, x_{min} *be the maximal and minimal entry of the principal eigenvector corresponding to* λ_1 *respectively, then*

$$\sqrt{\frac{\Delta}{\delta}} \leq \frac{x_{max}}{x_{min}}\,. \tag{2.5}$$

Lemma 2.7. *[7] Let* $d(v_i, v_j)$ *be the length of the shortest path joining the two vertices* v_i *and* v_j *of a graph* G*. If* $\left(x_{1_1}, x_{1_2}, ..., x_{1_p}\right)$ *is an eigenvector of* λ_1*, then for every* $v_i, v_j \in V(G)(i, j = 1, ..., p)$,

$$\lambda_1^{-d(v_i,v_j)} \leq \frac{x_{1_i}}{x_{1_j}}\,. \tag{2.6}$$

Below we present some bounds for x_{max} *and* $\frac{x_{max}}{x_{min}}$*. Unless otherwise specified by a graph* G *we refer a non-regular graph.*

3. Lower bound of x_{max}

In this section we present a lower bound for the maximal entry x_{max} of the principal eigenvector of the spectral radius λ_1.

Theorem 3.1. *Let* G *be a complete multipartite graph, then for the largest entry* x_{max} *of the principal eigenvector of* $A(G)$*, then*

$$\frac{(\lambda_1 - \delta)\sqrt{\frac{2q}{\lambda_1}}}{2q - p\delta} < x_{max}\,. \tag{3.1}$$

Proof. Let $V_1 = \left(x_{1_1}, x_{1_2}, ..., x_{1_p}\right)^T$ be the principal eigenvector corresponding to λ_1. Then by Rayleigh quotient $\lambda_1 = \dfrac{V_1^T A V_1}{V_1^T V_1}$, which gives $\lambda_1 = \displaystyle\sum_{v_i v_j \in E(G)} x_{1_i} x_{1_j}$.

Now

$$(\lambda_1 - \delta)\sum_{i=1}^{p} x_{1_i} = \sum_{i=1}^{p} d_i x_{1_i} - \delta \sum_{i=1}^{p} x_{1_i} \leq \sum_{i=1}^{p}(d_i - \delta)x_{max} = (2q - p\delta)x_{max}\ ,$$

since $\lambda_1 \displaystyle\sum_{i=1}^{p} x_{1_i} = \sum_{i=1}^{p} d_i x_{1_i}$. Thus

$$\frac{(\lambda_1 - \delta)\sum_{i=1}^{p} x_{1_i}}{2q - p\delta} \leq x_{max} \tag{3.2}$$

From (2.1) we get $W_1(G) = 2q = \sum_{i=1}^{p} \lambda_i C_i$. Since G is a complete multipartite graph, it contains only one positive eigenvalue [Lemma 2.2]. Therefore $2q < \lambda_1 C_1 = \lambda_1 \left(\sum_{i=1}^{p} x_{1_i}\right)^2$ which gives

$$\sqrt{\frac{2q}{\lambda_1}} < \left(\sum_{i=1}^{p} x_{1_i}\right) . \tag{3.3}$$

Combining (3.2) and (3.3) we get (3.1). □

Remark 3.1. *It seems that the bound* (3.1) *is better than the bound* (2.2) *in general. Below we compare the bounds* (2.2) *and* (3.1) *for the graphs in the fig.1.*

Example 1. *The values of* λ_1, x_{max} *and its lower bounds from* (2.2) *and* (3.1) *for the graphs in fig.1. are given in the following table.*

	λ_1	x_{max}	(2.2)	(3.1)
G_1	*1.7321*	*0.7072*	*0.5222*	*0.6813*
G_2	*3.2361*	*0.5257*	*0.4588*	*0.5249*
G_3	*3.7664*	*0.5093*	*0.4152*	*0.4631*

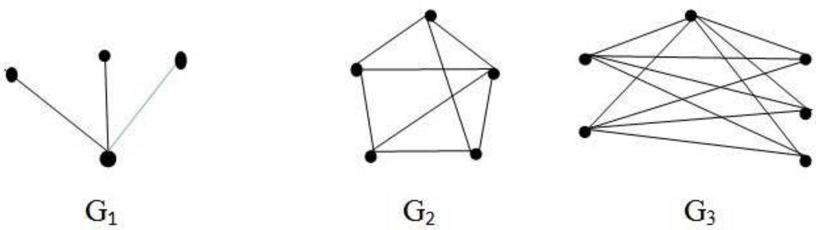


Fig.1.

Here we see that bound obtained from (3.1) *gives better results than the respective bound* (2.2) *for all the three graphs* G_1, G_2 *and* G_3.

Example 2. *Consider the graph* K_p/e *obtained by removing one edge e from a complete graph* K_p *of* p *vertices.The bound* (3.1) *gives better result for higher* p *in case of graphs* K_p/e*. The values of* p*,* λ_1 *and* x_{max} *for the graphs* $K_{50}/e, K_{30}/e$ *and* K_{10}/e *are compared below.*

	p	λ_1	x_{max}	(3.1)
K_{50}/e	50	48.9608	0.1415	0.1415
K_{30}/e	30	28.9353	0.1830	0.1830
K_{10}/e	10	8.8151	0.3220	0.3219

4. Lower bound of $\frac{x_{max}}{x_{min}}$

In this section we present a lower bound for the ratio of maximal entry x_{max} to the minimal entry x_{min} of the principal eigenvector of the graph G. Also we present some examples of graphs where our bound is better than the bounds from (2.5) and (2.6).

Theorem 4.1. *Let* G *be a graph of order* p *and* q *edges. If* x_{max} *and* x_{min} *are the maximum and minimum entries of the principal eigenvector corresponding to the spectral radius* λ_1 *then*

$$\frac{(p\lambda_1 - 2q)(\Delta - \bar{d})\sqrt{p(1+\lambda_1)}}{\alpha(p)(\Delta - \delta)(\Delta - \lambda_1)} + 1 < \frac{x_{max}}{x_{min}}\,, \tag{4.1}$$

where $\alpha(p) = p\left[\frac{p}{2}\right]\left(1 - \frac{1}{p}\left[\frac{p}{2}\right]\right)$ *and* $\bar{d} = \frac{\sum_{i=1}^{p} d_i}{p}$.

Proof. Let $V_1 = \left(x_{1_1}, x_{1_2}, ..., x_{1_p}\right)^T$ be the principal eigenvector corresponding to the spectral radius λ_1 of $A(G)$. Applying Rayleigh quotient, we get $\lambda_1 = \sum_{v_iv_j \in E(G)} x_{1_i}x_{1_j}$ which gives

$$\sqrt{1+\lambda_1} \leq \sum_{i=1}^{p} x_{1_i}\,. \tag{4.2}$$

Substituting the values a_i, b_i, a, A, b and B by $x_{1_i}, d_i, x_{min}, x_{max}, \delta$ and Δ respectively, where$(i = 1, 2, ..., p)$ in (2.4), we get

$$|p\sum_{i=1}^{p} x_{1_i}d_i - (\sum_{i=1}^{p} x_{1_i})(\sum_{i=1}^{p} d_i)| < \alpha(p)(x_{max} - x_{min})(\Delta - \delta)\,,$$

which gives $(p\lambda_1 - 2q)\sum_{i=1}^{p} x_{1_i} < \alpha(p)(x_{max} - x_{min})(\Delta - \delta)$, since G is a non-regular graph. Therefore, we can write

$$\frac{(p\lambda_1 - 2q)\sum_{i=1}^{p} x_{1_i}}{\alpha(p)(\Delta - \delta)x_{min}} + 1 < \frac{x_{max}}{x_{min}}\,. \tag{4.3}$$

Combining (2.3), (4.2) and (4.3) we get (4.1). □

Remark 4.1. *If x_{max} and x_{min} are the maximal and minimal entry of the principal eigenvector corresponding to λ_1, then*

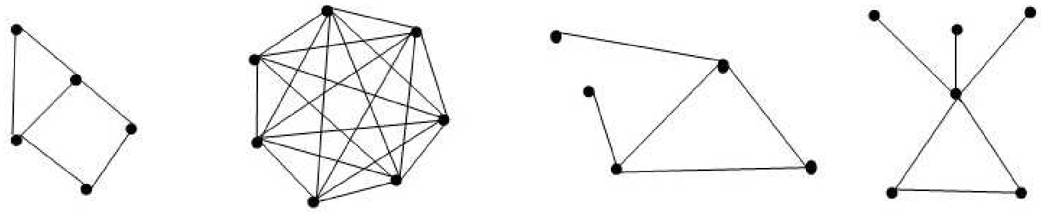
$$\frac{1}{\lambda_1^{d(x_{max}, x_{min})}} \leq \frac{1}{\lambda_1} \leq 1\,,$$

since $\lambda_1 \geq 1$. Thus for any graph G the bound from (4.1) *is always better than the bound from* (2.6).

Remark 4.2. *However, in general for some graphs the bound* (4.1) *is better than the bound* (2.5) *or for some other graphs it is not true*

Example 3. *Below we compare the bounds from* (2.5), (2.6) *and* (4.1) *for the graphs G_3, G_4, G_5 and G_6 in the fig.2*

	λ_1	$\frac{x_{max}}{x_{min}}$	(2.5)	(2.6)	(4.1)
G_3	*2.4812*	*1.4810*	*1.2247*	*0.4030*	*1.3265*
G_4	*5.7417*	*1.483*	*1.0954*	*0.1742*	*1.1214*
G_5	*2.3028*	*2.3028*	*1.7321*	*0.4343*	*1.7354*
G_6	*2.5141*	*2.5143*	*2.2361*	*0.3978*	*1.4748*



G_3 G_4 G_5 G_6

Fig.2. Bound (4.1) *is better than the bound* (2.6) *for all the four graphs* G_3, G_4, G_5 *and* G_6 *while it is better than the bound* (2.5) *in case of the graphs* G_3, G_4 *and* G_5*, but in case of the graph* G_6 *it is not true.*

Example 4. *It seems that for the graph of higher size,* $\alpha(p)$ *is high. In such cases the estimation* (4.1) *is not good. Also as the difference of the maximum and minimum degree increases in a graph the bound* (4.1) *starts to deviate rapidly from the actual value. We present the values of* $\lambda_1, \Delta - \delta, \alpha(p)$ *and* $\frac{x_{max}}{x_{min}}$ *and the lower bound from* (4.1) *for the path* P_{48} *[10] and the star graph* $K_{1,6}$ *[10] in the following table.*

	λ_1	$\Delta - \delta$	$\alpha(p) = p\left[\frac{p}{2}\right]\left(1 - \frac{1}{p}\left[\frac{p}{2}\right]\right)$	$\frac{x_{max}}{x_{min}}$	(4.1)
P_{48}	*1.9959*	*1*	*576*	*15.6512*	*1.3814*
$K_{1,6}$	*2.4495*	*5*	*12*	*2.4493*	*1.5088*

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