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TRAVELING WAVE EXACT SOLUTIONS FOR GENERAL SINE-GORDON EQUATION

SUBIN P. JOSEPH

ABSTRACT. Several variants of Sine-Gordon equations are used to describe various physical phenomena in nonlinear optics , propagation of fluxons in Josephson junctions charge density materials, and in many other fields. A general double Sine-Gordon equation is considered in this paper and several new exact solutions are derived. The solutions are derived in terms of Jacobi elliptic functions using traveling wave ansatz method.

1. INTRODUCTION

Different variants of Sine-Gordon equation are used to describe several physical phenomena in nonlinear optics , propogation of fluxons in Josephson junctions charge density materials, and in certain other fields [1–9, 11–15]. In this paper, we consider the general double Sine-Gordon equation given by

$$v_{tt} + \beta v_{xx} = \gamma \sin(nv) + \delta \sin(2nv), \tag{1.1}$$

where β, γ, δ are real constants and t and x are independent variables. The ordinary double Sine-Gordon equations is a special case of this general equation when $n = 1$. Only a few exact solutions are available for this general equation [12]. Traveling wave ansatz method is successfully applied to derive several exact solution for this general double Sin-Gordon equation in this paper. The method is described in the next section. Several new exact solutions are derived

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in the third section in terms of Jacobi elliptic functions and we conclude the paper with a discussion on the results obtained.

2. THE METHOD

Any partial diffferential equation of the form

$$P(v, v_t, v_x, v_{tt}, v_{tx}, v_{xx}, \cdots) = 0\,,$$

can be converted to an ODE of the form

$$O(v(u), v'(u), v''(u), \cdots) = 0,$$

by means of the transformation $u = at + bx$. Then, solving this ODE will lead to exact traveling wave solutions to the original PDE.

To convert the generalized double Sin-Gordon equation (1.1) in to an ODE, let

$$v(t,x) = \frac{2\arctan \chi(u)}{n}. \tag{2.1}$$

Then $\chi = \tan\left(\frac{nv}{2}\right)$,

$$\sin v = \frac{2\chi}{1+\chi^2},$$

$$v_{tt} + \beta v_{xx} = \frac{2\left(a^2+b^2\beta\right)\left(\chi^2\phi'' + \chi'' - 2\chi\chi'^2\right)}{n\left(\chi^2+1\right)^2}$$

and the double Sine-Gordon the equation (1.1) becomes

$$\begin{aligned} \chi\left(2\left(a^2+b^2\beta\right)\chi'^2 + n(\gamma+2\delta)\right) - \chi^2\left(a^2+b^2\beta\right)\chi'' - \left(a^2+b^2\beta\right)\chi'' \\ + n(\gamma-2\delta)\chi^3 = 0\,. \end{aligned} \tag{2.2}$$

We solve this ODE using different Jacobi elliptic function ansatz in the following section to derive several new traveling wave exact solutions for the double Sine-Gordon equation.

3. DIFFERENT EXACT SOLUTIONS

In this section several exact solutions are derived using Jacobi elliptic functions ansatz. Jacobi elliptic functions [10] are doubly periodic functions with modulus k, where $0 \leq k \leq 1$. For convenience we take $e = k^2$ with $0 \leq e \leq 1$ in this paper. The first ansatz we consider is

$$\chi(u) = A\ \mathrm{sn}(u). \tag{3.1}$$

Substitute this ansatz in the equation (2.2). On simplification we get a cubic polynomial in sn(u), with odd powers only. Equating to zero the coefficients of different powers of sn(u), we get the following non linear algebraic equations:

$$A\left(-2A^2\left(-a^2-b^2\beta\right)-e\left(-a^2-b^2\beta\right)+a^2+b^2\beta+\gamma n+2\delta n\right)=0,$$
$$A\left(A^2\left(-\left(a^2+b^2\beta\right)\right)+A^2e\left(-a^2-b^2\beta\right)-2e\left(a^2+b^2\beta\right)+A^2n(\gamma-2\delta)\right)=0.$$

Solving this system of non linear equations, we get the following sets of solutions

$$A=\pm\frac{\sqrt{\frac{\phi-\gamma(e+1)}{\gamma-2\delta}}}{\sqrt{2}},\quad a=\pm\frac{\sqrt{n(\phi-2\delta(e+1))-b^2\beta(e-1)^2}}{\sqrt{(e-1)^2}},$$

$$A=\pm\frac{\sqrt{\frac{\phi-\gamma(e+1)}{\gamma-2\delta}}}{\sqrt{2}},\quad a=\pm\frac{\sqrt{-b^2\beta(e-1)^2-n(2\delta(e+1)+\phi)}}{\sqrt{(e-1)^2}},$$

where $\phi = \sqrt{\gamma^2(e-1)^2+16\delta^2 e}$. Applying these solutions and using the equations (2.1) and (3.1), we get the following set of eight exact solutions for the Sine-Gordon equations:

(3.2)

$$v(t,x)=\frac{2}{n}\tan^{-1}\left(\pm\frac{\sqrt{\frac{\gamma(1+e)+\phi}{2\delta-\gamma}}}{\sqrt{2}}\mathrm{sn}\left(bx\pm\frac{\sqrt{n(\phi-2\delta(e+1))-b^2\beta(e-1)^2}}{\sqrt{(e-1)^2}}t\right)\right),$$

$$v(t,x)=\frac{2}{n}\tan^{-1}\left(\pm\frac{\sqrt{\frac{\phi-\gamma(e+1)}{\gamma-2\delta}}}{\sqrt{2}}\mathrm{sn}\left(bx\pm\frac{\sqrt{-b^2\beta(e-1)^2-n(2(e+1)\delta+\phi)}}{\sqrt{(e-1)^2}}t\right)\right).$$

To derive second family of exact solutions we consider the anstaz

$$\chi(u) = A\ \mathrm{cn}(u). \tag{3.3}$$

Substitute this ansatz in the equation (2.2). On simplification we get a cubic polynomial in cn(u), with odd powers only. Equating to zero the coefficients of different powers of $\mathrm{cn}(u)$, we get the following non linear algebraic equations

$$A\left(2A^2\left(a^2+b^2\beta\right)-2A^2e\left(a^2+b^2\beta\right)-e\left(a^2+b^2\beta\right)\right.$$
$$\left.+a^2(-e)+a^2+b^2\beta-b^2\beta e+\gamma n+2\delta n\right)=0$$
$$A\left(-2A^2\left(a^2+b^2\beta\right)+2A^2e\left(a^2+b^2\beta\right)+a^2A^2+e\left(a^2+b^2\beta\right)\right.$$
$$\left.+a^2e+A^2b^2\beta+A^2n(\gamma-2\delta)+b^2\beta e\right)=0.$$

Solving this system of non linear equations, we get the following four sets of solutions

$$A=\pm\frac{\sqrt{\frac{\gamma-2\gamma e-\psi}{(e-1)(\gamma-2\delta)}}}{\sqrt{2}},\quad a=\pm\sqrt{-b^2\beta-n(\delta(2-4e)+\psi)},$$

$$A=\pm\frac{\sqrt{\frac{\gamma-2\gamma e+\psi}{(e-1)(\gamma-2\delta)}}}{\sqrt{2}},\quad a=\pm\sqrt{n(\delta(4e-2)+\psi)-b^2\beta},$$

where $\psi=\sqrt{\gamma^2+16\delta^2(e-1)e}$. Applying these solutions and using the equations (2.1) and (3.3), we get the second set of eight exact solutions for the Sine-Gordon equations

$$v(t,x)=\frac{2}{n}\tan^{-1}\left(\pm\frac{\sqrt{\frac{\gamma-2\gamma e-\psi}{(e-1)(\gamma-2\delta)}}}{\sqrt{2}}\mathrm{cn}\left(bx+\pm\sqrt{-b^2\beta-n((2-4e)\delta+\psi)}t\right)\right),$$

$$v(t,x)=\frac{2}{n}\tan^{-1}\left(\pm\frac{\sqrt{\frac{\gamma-2\gamma e+\psi}{(e-1)(\gamma-2\delta)}}}{\sqrt{2}}\mathrm{cn}\left(bx\pm\sqrt{n(4e\delta-2\delta+\psi)-b^2\beta t}\right)\right). \tag{3.4}$$

4. DISCUSSION

In the previous section we derived two families of traveling wave exact solutions for generalized double Sine-Gordon equation (1.1), in terms of two different Jacobi elliptic functions $\mathrm{sn}(u)$ and $\mathrm{cn}(u)$. In the same way we can generate other exact solutions of double Sine-Gordon equation corresponding to the remaining ten Jacobi elliptic functions. So, we are getting twelve different families of exact solutions for this equation, each containing eight solutions in terms of one of the Jacobi elliptic functions.

It is also possible to derive periodic solutions from the derived new exact solutions. These solutions are obtained by letting the modulus of the Jacobi elliptic functions tends to zero in each of the solutions. For example, letting $e = 0$ in (3.2) and (3.4) we get the following periodic solutions,

$$\frac{2}{n}\arctan\left(\pm\sqrt{\frac{\gamma}{2\delta-\gamma}}\sin\left(bx \pm t\sqrt{n(\gamma-2\delta)-b^2\beta}\right)\right),$$

and

$$\frac{2}{n}\arctan\left(\pm\sqrt{\frac{\gamma}{2\delta-\gamma}}\cos\left(bx \pm t\sqrt{n(\gamma-2\delta)-b^2\beta}\right)\right),$$

respectively. Hence, we derived several doubly periodic and periodic traveling wave exact solutions for general double Sin-Gordon equation and the solutions obtained in terms of Jacobi elliptic functions are all new solutions in the literature. Also, all these solutions are validated by direct verification using computer algebra system.

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DEPARTMENT OF MATHEMATICS
GOVERNMENT ENGINEERING COLLEGE, WAYANAD
KERALA, INDIA
E-mail address: subinpj@gecwyd.ac.in