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NEW EXACT SOLUTIONS FOR GENERAL BOUSSINESQ EQUATION

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ABSTRACT. Boussinesq equation is a fourth order nonlinear partial differential equation and this equation represents the dynamics of several physical phenomena. In this work two different families of exact solutions for Boussinesq equation are derived. The Jacobi elliptic functions ansatz method and computer algebraic system are used to generate these new solutions.

1. INTRODUCTION

Nonlinear partial differential equations have great important in our modern world. The exact solution of nonlinear phenomena have important role in study of Physics, Engineering, Biological Sciences, Geological sciences and many other areas. The search of exact solutions to non linear partial differential equations by using computational methods is one of the major objective in non linear systems. In recent years many researchers have devoted their attention to find the exact solution of non linear partial differential equations using various methods. Some powerful method among these are tanh-method, tanh-coth method, sech-method, Exp-function method (G'/G)-Expansion method and Jacobi elliptic function method, Backlund transformation, Darboux transformation and inverse scattering method [1,6,8,9,11,14]. Practically there is no unified method that could be used to handle all types of nonlinear problems. Traveling wave

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solutions are widely used to derive exact solutions for most of the nonlinear partial differential equations.

Boussinesq equation is a fourth order nonlinear partial differential equation. The applications of Boussinesq equation plays an important role in modeling different process of wave propagation in two directions [3, 4]. Also the Boussinesq equation find its application to the dynamics of shallow water waves in coastal and ocean regions, the dynamics of thin inviscid layers and non-linear lattice waves [2–5, 7, 12, 13]. The Boussinesq equation with constant coefficients can be expressed in the following form

$$f_{tt} + \alpha f_{xx} + \beta\left(f f_x\right)_x + \gamma f_{xxxx} = 0, \tag{1.1}$$

where f is a function of the variables x and t and α, β and γ are arbitrary real parameters. The main purpose of this work is to find the exact solution of above Boussinesq equation from the idea of Jacobi elliptic functions ansatz method and computer algebraic system are used to generate these new solutions . The organization of the paper is as follows. In section 2, the method for obtaining exact solution is described using ansatz method. Later the new exact solutions of Boussinesq equation is generated in section 3.

2. THE ANSATZ METHOD

The solutions of the general Boussinesq equation (1.1) are derived using the traveling wave ansatz of the form

$$f(t,x) = g(ax+ct),$$

where a and c are arbitrary real parameters. Then the equation (1.1) becomes the ordinary differential equation

$$a^4\gamma g^{(4)}(u) + g''(u)\left(c^2 + a^2\beta g(u) + a^2\alpha\right) + a^2\beta g'(u)^2 = 0,$$

where $u = ax + ct$. To derive the required solutions ansatz function $g(u)$ will be taken to be in the rational form

$$g(u) = (A + B\ G(u)^2)/(A_1 + B_1\ G(u)^2), \tag{2.1}$$

where $G(u)$ is a function of the variable $u = at + bx$, and A, B, A_1 and B_1 are parameters to be determined. In this paper the functions $G(u)$ is taken to be Jacobi elliptic functions functions. These ansatz are substituted in the corresponding equation and simplified and derive the relations among the parameters to obtain the exact solutions. The computations are done using any of the computational algebra system. These solutions will lead to new exact solutions for different variants of Boussinesq equation.

3. Exact solutions

Jacobi elliptic functions [10] are doubly periodic functions with modulus k, where $0 \le k \le 1$. For convenience we take $e = k^2$ with $0 \le e \le 1$ in this paper. The first ansatz we consider is

$$g(u) = \frac{B\ sn^2(u) + A}{B_1\ sn^2(u) + A_1}. \tag{3.1}$$

Now we Substitute this in the equation (1.1) and simplify. Then this ansatz function will be a solution if the following algebraic equations are satisfied

$$B_1^2\left(12a^4B_2\gamma - a^2A_1\beta - B_1\left(-4a^4\gamma(e+1) + a^2\alpha + c^2\right)\right) = 0,$$

$$\begin{gathered}B_1\left(-8a^4B_1^2\gamma e^2 - 52a^4B_1^2\gamma e - 100a^4B_1B_2\gamma e - 8a^4B_1^2\gamma - 120a^4B_2^2\gamma - \right.\\ a^4B_1B_2\gamma + a^2A_1\beta\left(B_1(e+1) + 2B_2\right) - 3a^2A_2B_1\beta + 2a^2B_1^2\alpha e + 2a^2B_1^2\alpha + \\ \left.a^2B_1B_2\alpha + 2B_1^2c^2e + 2B_1^2c^2 + B_1B_2c^2\right) = 0,\end{gathered}$$

$$\begin{gathered}60a^4B_1^3\gamma e^2 + 88a^4B_1^2B_2\gamma e^2 + 60a^4B_1^3\gamma e + 220a^4B_1B_2^2\gamma e + 332a^4B_1^2B_2\gamma e + \\ 60a^4B_2^3\gamma + 220a^4B_1B_2^2\gamma + 88a^4B_1^2B_2\gamma + 4a^2A_2B_1^2\beta(e+1) - \\ a^2A_1\beta\left(3B_1^2e + 2B_2B_1(e+1) - 5B_2^2\right) - 3a^2B_1^3\alpha e + 2a^2B_1^2B_2\alpha e + \\ 5a^2B_1B_2^2\alpha + 2a^2B_1^2B_2\alpha - 3B_1^3c^2e + 2B_1^2B_2c^2e + 5B_1B_2^2c^2 + 2B_1^2B_2c^2 = 0,\end{gathered}$$

$$\begin{gathered}-60a^4B_1^3\gamma e^2 + a^2A_2\beta\left(-5B_1^2e + 2B_2B_1(e+1) + 3B_2^2\right) + \\ B_2^2\left(3B_2\left(-20a^4\gamma(e+1) + a^2\alpha + c^2\right) - 4a^2A_1\beta(e+1)\right) - \\ 2B_1B_2^2\left(2a^4\gamma\left(22e^2 + 83e + 22\right) + a^2\alpha(e+1) + c^2(e+1)\right) - \\ 5B_1^2B_2e\left(44a^4\gamma(e+1) + a^2\alpha + c^2\right) = 0,\end{gathered}$$

$$\begin{gathered}B_2\left(120a^4B_1^2\gamma e^2 - 2a^2A_2\beta\left(2B_1e + B_2(e+1)\right) + \right.\\ B_2\left(3a^2A_1\beta e - B_2\left(-2a^4\gamma\left(2e^2 + 13e + 2\right) + a^2\alpha(e+1) + c^2(e+1)\right)\right) \\ \left.-B_1B_2e\left(-100a^4\gamma(e+1) + a^2\alpha + c^2\right)\right) = 0,\end{gathered}$$

$$B_2^2e\left(-12a^4B_1\gamma e + a^2A_2\beta + B_2\left(-4a^4\gamma(e+1) + a^2\alpha + c^2\right)\right) = 0$$

Solving these equations simultaneously we get

$$A_1 = -\frac{B_1\left(-4a^4\gamma(e-2)+a^2\alpha+c^2\right)}{a^2\beta}, A_2 = \frac{B_1\left(4a^4\gamma(2e-1)+a^2\alpha+c^2\right)}{a^2\beta},$$
$$B_2 = -B_1,$$
$$A_1 = -\frac{B_1\left(4a^4\gamma(2e-1)+a^2\alpha+c^2\right)}{a^2\beta}, A_2 = \frac{B_1 e\left(-4a^4\gamma(e-2)+a^2\alpha+c^2\right)}{a^2\beta},$$
$$B_2 = B_1(-e)$$
$$A_1 = -\frac{12a^2B_2\gamma}{\beta}, A_2 = -\frac{B_2\left(-4a^4\gamma(e+1)+a^2\alpha+c^2\right)}{a^2\beta}, B_1 = 0$$

and

$$A_1 = -\frac{B_1\left(-4a^4\gamma(e+1)+a^2\alpha+c^2\right)}{a^2\beta}, A_2 = -\frac{12a^2B_1\gamma e}{\beta}, B_2 = 0.$$

Substituting these values in the ansatz function given by (3.1) we get the following family of exact solutions for the general Boussinesq equation

$$f_1(t,x) = \frac{-4a^4\gamma e + 8a^4\gamma + a^2\alpha - \left(4a^4\gamma(2e-1)+a^2\alpha+c^2\right)\mathrm{sn}(ct+ax)^2 + c^2}{a^2\beta\left(\mathrm{sn}(ct+ax)^2-1\right)},$$
$$f_2(t,x) = \frac{8a^4\gamma e - 4a^4\gamma + a^2\alpha - e\left(-4a^4\gamma(e-2)+a^2\alpha+c^2\right)\mathrm{sn}(ct+ax)^2 + c^2}{a^2\beta\left(e\,\mathrm{sn}(ct+ax)^2-1\right)},$$
$$f_3(t,x) = -\frac{\frac{12a^4\gamma}{\mathrm{sn}(ct+ax)^2} - 4a^4\gamma(e+1) + a^2\alpha + c^2}{a^2\beta},$$
$$f_4(t,x) = -\frac{12a^4\gamma e\,\mathrm{sn}(ct+ax)^2 - 4a^4\gamma e - 4a^4\gamma + a^2\alpha + c^2}{a^2\beta}.$$

To obtain another family of exact solutions, we consider the following ansatz form given by equation (2.1)

$$g(u) = \frac{A_2\ cn(u)^2 + A_1}{B_2\ cn(u)^2 + B_1}. \tag{3.2}$$

Substitute this in the general Boussinesq equation and simplify it. Then this ansatz is a solution to the equation (1.1) if the following algebraic equations are satisfied

$$B_1^2(e-1)\left(12a^4B_2\gamma(e-1)+a^2A_1\beta+B_1\left(4a^4\gamma(2e-1)+a^2\alpha+c^2\right)\right)=0,$$

$$\begin{gathered}-B_1\left(68a^4B_1^2\gamma e^2+120a^4B_2^2\gamma e^2+200a^4B_1B_2\gamma e^2-68a^4B_1^2\gamma e-240a^4B_2^2\gamma e-\right.\\ a^4B_1B_2\gamma e+a^4B_1^2\gamma+120a^4B_2^2\gamma+a^4B_1B_2\gamma-3a^2A_2B_1\beta(e-1)+\\ a^2A_1\beta\left(B_1(2e-1)+2B_2(e-1)\right)+4a^2B_1^2\alpha e+a^2B_1B_2\alpha e-2a^2B_1^2\alpha-\\ \left.a^2B_1B_2\alpha+B_1^2c^2e+B_1B_2c^2e-2B_1^2c^2-B_1B_2c^2\right)=0,\end{gathered}$$

$$B_2^2(-e)\left(12a^4B_1\gamma e+a^2A_2\beta+B_2\left(4a^4\gamma(2e-1)+a^2\alpha+c^2\right)\right)=0,$$

$$B_2\left(120a^4B_1^2\gamma e^2+2a^2A_2\beta\left(2B_1e+B_2(2e-1)\right)+\right.$$
$$B_2\left(2B_2\left(2a^4\gamma\left(17e^2-17e+2\right)+a^2\alpha(2e-1)+c^2(2e-1)\right)-3a^2A_1\beta e\right)+$$
$$\left.B_1B_2e\left(100a^4\gamma(2e-1)+a^2\alpha+c^2\right)\right)=0,$$

$$-60a^4B_1^3\gamma e^2+a^2A_2\beta\left(5B_1^2e+2B_2B_1(1-2e)-3B_2^2(e-1)\right)$$
$$+B_2^2\left(4a^2A_1\beta(2e-1)-3B_2(e-1)\left(20a^4\gamma(2e-1)+a^2\alpha+c^2\right)\right)+$$
$$2B_1B_2^2\left(-2a^4\gamma\left(127e^2-127e+22\right)+a^2\alpha(2e-1)+c^2(2e-1)\right)+$$
$$5B_1^2B_2e\left(44a^4\gamma(1-2e)+a^2\alpha+c^2\right)=0,$$

$$120a^4B_1^3\gamma e^2+60a^4B_2^3\gamma e^2+440a^4B_1B_2^2\gamma e^2+508a^4B_1^2B_2\gamma e^2-60a^4B_1^3\gamma e$$
$$-120a^4B_2^3\gamma e-660a^4B_1B_2^2\gamma e-508a^4B_1^2B_2\gamma e+60a^4B_2^3\gamma+220a^4B_1B_2^2\gamma+$$
$$88a^4B_1^2B_2\gamma+4a^2A_2B_1^2\beta(1-2e)+a^2A_1\beta\left(3B_1^2e+2B_2B_1(2e-1)-\right.$$
$$\left.B_2^2(e-1)\right)+3a^2B_1^3\alpha e-5a^2B_1B_2^2\alpha e-4a^2B_1^2B_2\alpha e+5a^2B_1B_2^2\alpha+$$
$$2a^2B_1^2B_2\alpha+3B_1^3c^2e-5B_1B_2^2c^2e-4B_1^2B_2c^2e+5B_1B_2^2c^2+2B_1^2B_2c^2=0.$$

Solving this system of equations we get the following relationship among the parameters

$$A_1=\frac{B_2\left(-4a^4\gamma(e-2)+a^2\alpha+c^2\right)}{a^2\beta},\ A_2=-\frac{B_2\left(-4a^4\gamma(e+1)+a^2\alpha+c^2\right)}{a^2\beta},$$
$$B_1=-B_2,$$
$$A_1=\frac{B_2(e-1)\left(-4a^4\gamma(e+1)+a^2\alpha+c^2\right)}{a^2\beta e},\ A_2=-\frac{B_2\left(-4a^4\gamma(e-2)+a^2\alpha+c^2\right)}{a^2\beta},$$
$$B_1=-\frac{B_2(e-1)}{e}$$
$$A_1=-\frac{B_1\left(4a^4\gamma(2e-1)+a^2\alpha+c^2\right)}{a^2\beta},\ A_2=\frac{12a^2B_1\gamma e}{\beta},\ B_2=0$$

and

$$A_1=\frac{12a^2B_2\gamma(e-1)}{\beta},\ A_2=-\frac{B_2\left(4a^4\gamma(2e-1)+a^2\alpha+c^2\right)}{a^2\beta},\ B_1=0.$$

So the new family of exact solution for the general Boussinesq equation is obtained by substituting these values in (3.2), which gives

$$
\begin{aligned}
f_5(t,x) =& \frac{-4a^4\gamma e + 8a^4\gamma + a^2\alpha - \left(-4a^4\gamma(e+1) + a^2\alpha + c^2\right)\mathrm{cn}(ct+ax)^2 + c^2}{a^2\beta\left(\mathrm{cn}(ct+ax)^2 - 1\right)},\\
f_6(t,x) =& \frac{(e-1)\left(-4a^4\gamma(e+1) + a^2\alpha + c^2\right)}{a^2\beta\left(e\mathrm{cn}(ct+ax)^2 - e + 1\right)}\\
&- \frac{e\left(-4a^4\gamma(e-2) + a^2\alpha + c^2\right)\mathrm{cn}(ct+ax)^2}{a^2\beta\left(e\mathrm{cn}(ct+ax)^2 - e + 1\right)},\\
f_7(t,x) =& -\frac{-12a^4\gamma e\mathrm{cn}(ct+ax)^2 + 8a^4\gamma e - 4a^4\gamma + a^2\alpha + c^2}{a^2\beta},\\
f_8(t,x) =& -\frac{-\frac{12a^4\gamma(e-1)}{\mathrm{cn}(ct+ax)^2} + 4a^4\gamma(2e-1) + a^2\alpha + c^2}{a^2\beta}.
\end{aligned}
$$

4. CONCLUSION

In this paper we have generated two families of new exact solutions for general Boussinesq equation (1.1). The ansatz method and computer algebra system are used for deriving solutions. The solutions are obtained in terms of Jacobi elliptic functions sn(u) and cn(u). From all these solutions we can generate periodic solutions and hyperbolic solutions as the limiting case of Jacobian Eliptic functions when e=0 or 1. We can generate exact solutions for the different variants of Boussinesq equation by a suitable choice for the values of the parameters in the solution. It is to be noted that the solutions obtained here are valid for arbitrary values of a and b where $u = at + bx$. These solutions can be used for analyzing the different physical phenomena suitably represented by these equations such as the dynamics of shallow water waves in coastal and ocean regions, the dynamics of thin inviscid layers and non-linear lattice waves. The accuracy of several approximate methods used to derive numerical solutions for Boussinesq equation can also be tested using these solutions.

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