ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.4, 2333–2343
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.4.95 Spec. Issue on NCFCTA-2020

EXISTENCE RESULTS FOR IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS WITH FRACTIONAL BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we examine the existence of solutions for implicit FDE's with fractional boundary conditions. To prove the existence results by applying fixed point theorems and continuous on parameters and functions. Finally an example is included to show the applicability of our results.

1. INTRODUCTION

The fundamental of the fractional calculus and FDE's has been proved by applying importance in the modeling of many development in various fields of engineering, medicine, chemistry, physics, economics and signal processing. For more details on this theory and on its applications, it is to be refered in [7, 8, 13–15].

In [4] M. Benchohra and J. E. Lazreg have given an investigation the IFDE's and [12] K. D. Kucche, J. J. Nieto and V. Venktesh have given an investigation the nonlnear IFDE's and continuous dependence. Recently we refer the [9] S. K. Ntouyas and J. Tariboon have considered the FBVP with multiple order of fractional derivatives and integral by applied the single-valued case using Sadovski's fixed point theorem. The reader for further identification and clarification need to refer the papers of [1–3, 5, 6, 10, 11, 16, 17].

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2010 *Mathematics Subject Classification.* 34A09, 34A08,26A33,34A12,34B10.

Key words and phrases. Implicit fractional differential equations, fractional boundary conditions, existence, fixed point theorems.

Motivated by above research papers, we study the the existence of solutions for implicit FDE's with fractional boundary conditions of the forms

$$(1.1)\qquad {}^cD^{\alpha}x(t) = f(t, x(t), {}^cD^{\alpha}x(t)), \quad t \in J := (0,T), \quad 1 < \alpha \leq 2,$$

$$(1.2)\qquad x(0) = 0, \quad \lambda D^{\beta_1}x(T) + (1-\lambda)D^{\beta_2}x(T) = \beta_3,$$

where D^{ϕ} is the Caputo fractional derivative of order $\phi \in \{\alpha, \beta_1, \beta_2\}$ such that $1 < \alpha \leq 2$, $0 < \beta_1, \beta_2 < \alpha$, $\beta_3 \in \mathbb{R}$, $0 \leq \lambda \leq 1$ is given constant and $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function.

In this paper is planned as shades. Section 2 has definitions and elementary results of the fractional calculus. In section 3, implicit FDE's with fractional boundary conditions are proved the theorems on the existence results by applying fixed point theorems, continuous dependence on parameters and function involved in the equations. In section 4, an illustrative example is provided in support of the results of a problem (1.1) and (1.2).

2. Preliminaries

In this section, the most important basic concepts and lemma are stated.

Definition 2.1. *For a function $h \in AC^n(J)$, the Caputo's fractional-order derivative of order α is defined by $({}^cD_0^{\alpha})(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds$, where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of the real number α.*

Definition 2.2. *A function $x \in PC'(J,\mathbb{R})$ is said to be a solution of the problem (1.1), if $x(t) = x_k(t)$ for $t \in (t_k, t_{k+1})$ and $x_k \in C([0 = t_0 < t_1 < ... < t_m < t_{m+1} = T], \mathbb{R})$ satisfies ${}^cD^{\alpha}x_k(t) = f(t, x_k(t), {}^cD^{\alpha}x_k(t))$, almost everywhere on $(0, t_{k+1})$ with the restriction of $x_k(t)$ on $[0, t_k)$ is just $x_{k-1}(t)$ and the conditions $\Delta x(t_k) = y_k,\ \Delta x'(t_k) = \bar{y}_k,\ y_k, \bar{y}_k \in \mathbb{R}\ k = 1, 2, ..., m$ with $x(0) = 0,\ x'(1) = 0$.*

Lemma 2.1. *For $\alpha > 0$, the general solution of the FDE's ${}^cD^{\alpha}x(t) = 0$ is given by $x(t) = c_0 + c_1t + \ldots + c_{n-1}t^{n-1}$, where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ ($n = [\alpha]+1$).*

In view of Lemma 2.1, it follows that $I^{\alpha}\, {}^cD^{\alpha}x(t) = x(t) + c_0 + c_1t + \ldots + c_{n-1}t^{n-1}$, for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ $(n = [\alpha]+1)$.

Lemma 2.2. *The boundary value problem*

$$(2.1)\qquad \begin{aligned} &D^{\alpha}x(t) = \omega(t), \quad t \in (0,T),\\ &x(0) = 0, \quad \lambda D^{\beta_1}x(T) + (1-\lambda)D^{\beta_2}x(T) = \beta_3, \end{aligned}$$

is equivalent to the integral equation

$$
\begin{aligned}
x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\omega(s)ds + \frac{t}{\Lambda_1}\Big(\beta_3 \\
- \frac{\lambda}{\Gamma(\alpha-\beta_1)} \int_0^T (T-s)^{\alpha-\beta_1+1}\omega(s)ds \\
- \frac{1-\lambda}{\Gamma(\alpha-\beta_2)} \int_0^T (T-s)^{\alpha-\beta_2+1}\omega(s)ds\Big), \quad t \in J := [0,T],
\end{aligned} \tag{2.2}
$$

where the non zero constant Λ_1 *is defined by* $\Lambda_1 = \frac{\lambda T^{1-\beta_1}}{\Gamma(2-\beta_1)} + \frac{(1-\lambda)T^{1-\beta_2}}{\Gamma(2-\beta_2)}$.

Proof. From the first equation of (2.1), we have $D^\alpha x(t) = \omega(t), t \in J$. We obtain $x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\omega(s)ds + C_1 + C_2 t$, for C_1, $C_2 \in \mathbb{R}$. The first boundary condition of (2.1) implies that $C_1 = 0$. Hence

$$
x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\omega(s)ds + C_2 t. \tag{2.3}
$$

Applying the Caputo fractional derivative of order $\psi \in \{\beta_1, \beta_2\}$ such that $0 < \psi < \alpha - \beta$ to (2.3), we have

$$
D^\psi x(t) = \frac{1}{\Gamma(\alpha-\psi)} \int_0^t (t-s)^{\alpha-\psi-1}\omega(s)ds + C_2 \frac{1}{\Gamma(2-\psi)} t^{1-\psi}.
$$

Substituting the values $\psi = \beta_1$ and $\psi = \beta_2$ to the above relation and using the second condition of (2.1), we obtain

$$
\begin{aligned}
\beta_3 = \frac{\lambda}{\Gamma(\alpha-\beta_1)} \int_0^T (T-s)^{\alpha-\beta_1+1}\omega(s)ds + \frac{\lambda T^{1-\beta_1}}{\Gamma(2-\beta_1)} C_2 \\
+ \frac{1-\lambda}{\Gamma(\alpha-\beta_2)} \int_0^T (T-s)^{\alpha-\beta_2+1}\omega(s)ds + \frac{(1-\lambda)T^{1-\beta_2}}{\Gamma(2-\beta_2)} C_2,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
C_2 = \frac{1}{\Lambda_1}\Big[\beta_3 - \frac{\lambda}{\Gamma(\alpha-\beta_1)} \int_0^T (T-s)^{\alpha-\beta_1+1}\omega(s)ds \\
- \frac{1-\lambda}{\Gamma(\alpha-\beta_2)} \int_0^T (T-s)^{\alpha-\beta_2+1}\omega(s)ds\Big].
\end{aligned}
$$

Substituting the value of the constant C_2 in (2.3), we deduce the integral equation (2.2). The converse follows by direct computation. This completes the proof. □

3. Main Results

To prove the existence and uniqueness results we need the following assumptions: (A_1) The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. (A_2) There exists constants $K > 0$ and $0 < L < 1$ such that $|f(t,u,v) - f(t,u_1,v_1)| \leq K|u-u_1| + L|v-v_1|$, for any $u, v, u_1, v_1 \in \mathbb{R}, t \in J$. The two fractional boundary value problem (1.1)-(1.2) is equivalent to the integral equation

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s), {}^c D^\alpha x(s))ds \\ &+ \frac{t}{\Lambda_1}\Big(\beta_3 - \frac{\lambda}{\Gamma(\alpha-\beta_1)}\int_0^T (T-s)^{\alpha-\beta_1+1} f(s,x(s),{}^c D^\alpha x(s))ds \\ &- \frac{(1-\lambda)}{\Gamma(\alpha-\beta_2)}\int_0^T (T-s)^{\alpha-\beta_2+1} f(s,x(s),{}^c D^\alpha x(s))ds\Big), \quad t \in J, \end{aligned}$$

where the non zero constant Λ_1 is defined by

$$\Lambda_1 = \frac{\lambda T^{1-\beta_1}}{\Gamma(2-\beta_1)} + \frac{(1-\lambda)T^{1-\beta_2}}{\Gamma(2-\beta_2)}.$$

Theorem 3.1. *Assume that* (A_1) *and* (A_2) *are holds. If*

$$\left[\frac{T^\alpha}{\Gamma(\alpha+1)} - \frac{T}{\Lambda_1}\left\{\frac{\lambda T^{\alpha-\beta_1+1}}{\Gamma(\alpha-\beta_1+1)} + \frac{(1-\lambda)T^{\alpha-\beta_2+1}}{\Gamma(\alpha-\beta_2+1)}\right\}\right]\frac{K}{(1-L)} < 1,$$

then there exists a unique solution for (1.1)-(1.2) *on* J.

Proof. Let $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$ be a closed bounded and convex subset of $\mathcal{C}$, where r is a fixed constant. Consider the operator $\ominus : \mathcal{C} \to \mathcal{C}$ defined by

$$\ominus y(t) = I^\alpha g(t) + \frac{t}{\Lambda_1}\left[\gamma_3 - I^{\alpha-\beta_1} g_1(t) - I^{\alpha-\beta_2} g_2(t)\right], \tag{3.1}$$

where $g(t) = f(t, x(t), g(t)), g_1(t) = f(t, x(t), g_1(t)), g_2(t) = f(t, x(t), g_2(t))$, $g, g_1, g_2 \in C(J, \mathbb{R})$. Clearly, the fixed points of operator $\ominus$ is solution of problem (1.1)-(1.2). Let $x_1, x_2 \in C(J,\mathbb{R})$. Then,

$$\begin{aligned} |(\ominus x_1)(t) - (\ominus x_2)(t)| &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|(g(s)-h(s))|ds \\ &- \frac{t}{\Lambda_1}\Big(\frac{\lambda}{\Gamma(\alpha-\beta_1)}\int_0^T (T-s)^{\alpha-\beta_1+1}|(g_1(s)-h_1(s))|ds \\ &+ \frac{(1-\lambda)}{\Gamma(\alpha-\beta_2)}\int_0^T (T-s)^{\alpha-\beta_2+1}|(g_2(s)-h_2(s))|ds\Big), \end{aligned} \tag{3.2}$$

where $g, h, g_1, g_2, h_1, h_2 \in C(J, \mathbb{R})$ be such that

$$g(t) = f(t, x_1(t), g(t)),\ g_1(t) = f(t, x_1(t), g_1(t)),$$
$$g_2(t) = f(t, x_1(t), g_2(t)),\ h(t) = f(t, x_1(t), h(t)),$$
$$h_1(t) = f(t, x_1(t), h_1(t)),\ h_2(t) = f(t, x_1(t), h_2(t)).$$

By hypothesis (A_2), we have

$$|(g(t) - h(t))| \le K|x_1(t) - x_2(t)| + L|x_1(t) - x_2(t)| \le \frac{K}{1-L}|x_1(t) - x_2(t)|$$

$$|(g_1(t) - h_1(t))| \le \frac{K}{1-L}|x_1(t) - x_2(t)|$$

and

$$|(g_2(t) - h_2(t))| \le \frac{K}{1-L}|x_1(t) - x_2(t)|.$$

The equation (3.2) implies

$$|(\ominus x_1)(t) - (\ominus x_2)(t)| \le \frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}||x_1 - x_2||_\infty$$
$$- \frac{t}{\Lambda_1}\Big(\frac{\lambda K T^{\alpha-\beta_1+1}}{(1-L)\Gamma(\alpha-\beta_1+1)} + \frac{(1-\lambda)KT^{\alpha-\beta_2+1}}{(1-L)\Gamma(\alpha-\beta_2+1)}\Big)||x_1 - x_2||_\infty.$$

Thus

$$||\ominus x_1 - \ominus x_2||_\infty \le \left[\frac{T^\alpha}{\Gamma(\alpha+1)} - \frac{T}{\Lambda_1}\left[\frac{\lambda T^{\alpha-\beta_1+1}}{\Gamma(\alpha-\beta_1+1)} + \frac{(1-\lambda)T^{\alpha-\beta_2+1}}{\Gamma(\alpha-\beta_2+1)}\right]\right]$$
$$\cdot \frac{K}{(1-L)}||x_1 - x_2||_\infty.$$

By (3.1), the operator $\ominus$ is a continuous. Hence by Banach's contraction principle, $\ominus$ has a unique fixed point which is a unique solution of the problem (1.1)-(1.2). □

4. Continuous on parameters and functions

$$^cD^\alpha x_1(t) = f(t, x_1(t), ^cD^\alpha x_1(t), \delta_1), \quad t \in (0,T), \quad 1 < \alpha \le 2$$
$$x_1(0) = 0, \quad \lambda D^{\beta_1} x_1(T) + (1-\lambda) D^{\beta_2} x_1(T) = \beta_3, \tag{4.1}$$

and

$$
\begin{aligned}
&{}^cD^{\alpha}x_2(t) = f(t, x_2(t), {}^cD^{\alpha}x_2(t), \delta_2), \quad t \in (0,T), \quad 1 < \alpha \leq 2 \\
&x_2(0) = 0, \quad \lambda D^{\beta_1}x_2(T) + (1-\lambda)D^{\beta_2}x_2(T) = \beta_3,
\end{aligned} \tag{4.2}
$$

where D^{α_1} is the Caputo fractional derivative of order $\alpha_1 \in \{\alpha, \beta_1, \beta_2\}$ such that $1 < \alpha \leq 2$, $0 < \beta_1, \beta_2 < \alpha$, $\beta_3 \in \mathbb{R}$, $0 \leq \lambda \leq 1$ is given constant, δ_1, δ_2 are real parameters and $f : J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We need the assumptions and lemma to prove the dependence of solution of implicit fractional differential equations on parameters: (A_3) There exists $h \in C(J, \mathbb{R})$, $l \in L_1[0,T]$ and $M \in (0,1)$ the continuous function f satisfies $|f(t,x,y,\delta_1) - f(t,x_1,y_1,\delta_2)| \leq h(t)|x - x_1| + M|y - y_1|$ and $|g(t,x,y,\delta_1) - g(t,x_1,y_1,\delta_2)| \leq l(t)|\delta_1 - \delta_2|$.

Lemma 4.1. *let $V : [0,T] \to [0,+\infty)$ be a real function and $W(.)$ is nonnegative, locally integrable function on $[0,T]$. Assume that there is a constant $a > 0$ such that for $0 < \alpha \leq 2$. $V(t) \leq W(t) + a\int_0^t (t-s)^{-\alpha}V(s)ds$. Then, there exists a constant $K = K(\alpha)$ such that $V(t) \leq W(t) + Ka\int_0^t (t-s)^{-\alpha}W(s)ds$ for any $t \in [0,T]$.*

Theorem 4.1. *Let $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy (A_3). If $x_1(t)$ and $x_2(t)$ are the solutions of (4.1) and (4.2) respectively, then*

$$
\begin{aligned}
|x_1(s) - x_2(s)| \leq |\delta_1 - \delta_2| \Big[& I^{\alpha}l(t) + \frac{KH}{(1-M)\Gamma(\alpha)} I^{\alpha}(I^{\alpha}l(t)) \\
& + \frac{t}{\Lambda_1}\Big(I^{\alpha-\beta_1}l(t) + I^{\alpha-\beta_2}l(t) + \frac{\lambda KH}{(1-M)\Gamma(\alpha-\beta_1)} I^{\alpha-\beta_1}(I^{\alpha-\beta_1}l(t)) \\
& + \frac{(1-\lambda)KH}{(1-M)\Gamma(\alpha-\beta_2)} I^{\alpha-\beta_2}(I^{\alpha-\beta_2}l(t)) \Big) \Big], t \in [0,T],
\end{aligned}
$$

where K is a constant depending on α and $H = \max\{h(t), t \in [0,T]\}$.

Proof. Let $x_1(t)$ and $x_2(t)$ be the solution of (4.1) and (4.2) respectively, then

$$
\begin{aligned}
&{}^cD^{\alpha}x_1(t) = f(t, x_1(t), {}^cD^{\alpha}x_1(t), \delta_1), \quad t \in (0,T), \quad 1 < \alpha \leq 2, \\
&x_1(0) = 0, \quad \lambda D^{\beta_1}x_1(T) + (1-\lambda)D^{\beta_2}x_1(T) = \beta_3,
\end{aligned}
$$

and

$$
\begin{aligned}
&{}^cD^{\alpha}x_2(t) = f(t, x_2(t), {}^cD^{\alpha}x_2(t), \delta_2), \quad t \in (0,T), \quad 1 < \alpha \leq 2, \\
&x_2(0) = 0, \quad \lambda D^{\beta_1}x_2(T) + (1-\lambda)D^{\beta_2}x_2(T) = \beta_3.
\end{aligned}
$$

Implies

$$\begin{aligned}x_1(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x_1(s), {}^c D^\alpha x_1(s), \delta_1)ds \\ &+ \frac{t}{\Lambda_1}\Big(\beta_3 - \frac{\lambda}{\Gamma(\alpha-\beta_1)}\int_0^T (T-s)^{\alpha-\beta_1+1} f(s, x_1(s), {}^c D^\alpha x_1(s), \delta_1)ds \\ &- \frac{(1-\lambda)}{\Gamma(\alpha-\beta_2)}\int_0^T (T-s)^{\alpha-\beta_2+1} f(s, x_1(s), {}^c D^\alpha x_1(s), \delta_1)ds\Big), \quad t \in J,\end{aligned}$$

and

$$\begin{aligned}x_2(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x_2(s), {}^c D^\alpha x_2(s), \delta_2)ds \\ &+ \frac{t}{\Lambda_1}\Big(\beta_3 - \frac{\lambda}{\Gamma(\alpha-\beta_1)}\int_0^T (T-s)^{\alpha-\beta_1+1} f(s, x_2(s), {}^c D^\alpha x_2(s), \delta_2)ds \\ &- \frac{(1-\lambda)}{\Gamma(\alpha-\beta_2)}\int_0^T (T-s)^{\alpha-\beta_2+1} f(s, x_2(s), {}^c D^\alpha x_2(s), \delta_2)ds\Big), \quad t \in J.\end{aligned}$$

$$\begin{aligned}&|x_1(s) - x_2(s)| \\ &= |\delta_1 - \delta_2| I^\alpha l(t) + \frac{H}{(1-M)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x_1(s) - x_2(s)|ds \\ &+ \frac{t}{\Lambda_1}\Big(|\delta_1 - \delta_2| I^{\alpha-\beta_1} l(t) + |\delta_1 - \delta_2| I^{\alpha-\beta_2} l(t) \\ &+ \frac{\lambda H}{(1-M)\Gamma(\alpha-\beta_1)}\int_0^T (T-s)^{\alpha-\beta_1+1}|x_1(s) - x_2(s)|ds \\ &+ \frac{(1-\lambda)H}{(1-M)\Gamma(\alpha-\beta_2)}\int_0^T (T-s)^{\alpha-\beta_2+1}|x_1(s) - x_2(s)|ds\Big)\end{aligned} \tag{4.3}$$

By lemma 4.1, the equation (4.3) implies that

$$\begin{aligned}|x_1(s) - x_2(s)| \le |\delta_1 - \delta_2| \Big[& I^\alpha l(t) + \frac{KH}{(1-M)\Gamma(\alpha)} I^\alpha(I^\alpha l(t)) \\ &+ \frac{t}{\Lambda_1}\Big(I^{\alpha-\beta_1} l(t) + I^{\alpha-\beta_2} l(t) + \frac{\lambda KH}{(1-M)\Gamma(\alpha-\beta_1)} I^{\alpha-\beta_1}(I^{\alpha-\beta_1} l(t)) \\ &+ \frac{(1-\lambda)KH}{(1-M)\Gamma(\alpha-\beta_2)} I^{\alpha-\beta_2}(I^{\alpha-\beta_2} l(t))\Big)\Big], t \in [0,T].\end{aligned}$$

□

Next result, proves the Continuous dependence of solution of IFDE's (1.1)-(1.2) on the function involved in right hand side of equation (1.1)-(1.2).

$$(4.4)\qquad \begin{aligned} &{}^cD^{\alpha}y(t) = \tilde{f}(t, y(t), {}^cD^{\alpha}y(t)), \quad t \in (0,T), \quad 1 < \alpha \leq 2 \\ &y(0) = 0, \quad \lambda D^{\beta_1}y(T) + (1-\lambda)D^{\beta_2}y(T) = \tilde{\beta}_3, \end{aligned}$$

where $\tilde{f} : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\tilde{\beta}_3 \in \mathbb{R}$.

Theorem 4.2. *Suppose that f in (1.1) Satisfies the hypothesis: there exists $q \in C[J, \mathbb{R}]$ and $L \in (0,1)$ such that $|f(t,x,y) - f(t,x_1,y_1)| \leq q(t)|x - x_1| + L|y - y_1|$ where $Q = \max\{q(t), t \in [0,T]\}$. Further suppose, for arbitirarily small constant ϵ, $\delta > 0$ that $|f(t, x(t), D^{\alpha}x(t)) - \tilde{f}(t, y(t), D^{\alpha}y(t))| \leq \epsilon$ and $|\beta_3 - \tilde{\beta}_3| < \delta$, $t \in [0,T]$. Then the solution $x(t)$ of (1.1) depends continuously on the functions involved in right hand side of equation (1.1).*

Proof. Let $x(t)$ and $y(t)$ be the solution of (1.1) and (4.4) respectively, then

$$\begin{aligned} &{}^cD^{\alpha}x(t) = f(t, x(t), {}^cD^{\alpha}x(t)), \quad t \in (0,T), \quad 1 < \alpha \leq 2 \\ &x(0) = 0, \quad \lambda D^{\beta_1}x(T) + (1-\lambda)D^{\beta_2}x(T) = \beta_3, \end{aligned}$$

and

$$\begin{aligned} &{}^cD^{\alpha}y(t) = \tilde{f}(t, y(t), {}^cD^{\alpha}y(t)), \quad t \in (0,T), \quad 1 < \alpha \leq 2 \\ &y(0) = 0, \quad \lambda D^{\beta_1}y(T) + (1-\lambda)D^{\beta_2}y(T) = \tilde{\beta}_3, \end{aligned}$$

implies

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), {}^cD^{\alpha}x(s))ds \\ & + \frac{t}{\Lambda_1}\Big(\beta_3 - \frac{\lambda}{\Gamma(\alpha-\beta_1)} \int_0^T (T-s)^{\alpha-\beta_1+1} f(s, x(s), {}^cD^{\alpha}x(s))ds \\ & - \frac{(1-\lambda)}{\Gamma(\alpha-\beta_2)} \int_0^T (T-s)^{\alpha-\beta_2+1} f(s, x(s), {}^cD^{\alpha}x(s))ds\Big), \quad t \in J, \end{aligned}$$

and

$$\begin{aligned} y(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \tilde{f}(s, y(s), {}^cD^{\alpha}y(s))ds \\ & + \frac{t}{\Lambda_1}\Big(\tilde{\beta}_3 - \frac{\lambda}{\Gamma(\alpha-\beta_1)} \int_0^T (T-s)^{\alpha-\beta_1+1} \tilde{f}(s, y(s), {}^cD^{\alpha}y(s))ds \\ & - \frac{(1-\lambda)}{\Gamma(\alpha-\beta_2)} \int_0^T (T-s)^{\alpha-\beta_2+1} \tilde{f}(s, y(s), {}^cD^{\alpha}y(s))ds\Big), \quad t \in J. \end{aligned}$$

By using the hypothesis and

$$|D^{\alpha}(x(t)-y(t))| \leq |f(t,x(t),^{c}D^{\alpha}x(t)) - \tilde{f}(t,y(t),^{c}D^{\alpha}y(t))|$$
$$\leq \frac{q(t)}{(1-L)}|x(t)-y(t)| + \frac{\epsilon}{(1-L)}.$$

By lemma 4.1,

(4.5)

$$\begin{aligned}
&|x(t)-y(t)| \\
&\leq \epsilon\left[\frac{1}{1-L}\right]\left\{\frac{1}{\Gamma(\alpha+1)}t^{\alpha} + \frac{KQ}{(1-L)\Gamma(2\alpha+1)}t^{2\alpha}\right\} \\
&\quad - \frac{t}{\Lambda_1}\Bigg(\delta\left\{1 + \frac{KQ\lambda}{(1-L)\Gamma(\alpha-\beta_1+1)}T^{\alpha-\beta_1} + \frac{KQ(1-\lambda)}{(1-L)\Gamma(\alpha-\beta_2+1)}T^{\alpha-\beta_2}\right\} \\
&\quad + \epsilon\left[\frac{1}{1-L}\right]\Bigg(\frac{KQ\lambda^2}{(1-L)\Gamma(2\alpha-\beta_1+1)}T^{2\alpha-\beta_1} + \frac{KQ(1-\lambda)^2}{(1-L)\Gamma(2\alpha-\beta_2+1)}T^{2\alpha-\beta_2} \\
&\quad + \frac{\lambda T^{\alpha-\beta_1}}{\Gamma(\alpha-\beta_1+1)} + \frac{(1-\lambda)T^{\alpha-\beta_2}}{\Gamma(\alpha-\beta_2+1)}\Bigg)\Bigg).
\end{aligned}$$

From the equation (4.5), it follows that the solution $x(t)$ of (1.1) depends continuouly on the functions involved in right hand side of eqaution (1.1). For $\epsilon = 0$ in the inequality (4.5) gives continuous dependence of solutions on boundary conditions. We also note that as $\epsilon, \delta > 0$ were arbitrary, by taking $\epsilon, \delta \to 0^{+}$, we have $x \to y$ where $x : [0,T] \to \mathbb{R}$ and $y : [0,T] \to \mathbb{R}$ are the solution of (1.1) and (4.4) respectively. □

Example 1. *Consider the implicit FDE's with fractional boundary conditions of the form*

$$(4.6) \qquad {}^{c}D^{\frac{10}{7}}x(t) = \frac{1}{10(1+|x(t)|+|{}^{c}D^{\frac{10}{7}}x(t)|)}, \quad t \in (0,T), \quad 1 < \alpha \leq 2,$$

$$(4.7) \qquad x(0) = 0, \quad \frac{8}{20}D^{\frac{6}{14}}x(1) + \frac{3}{5}D^{\frac{4}{17}}x(1) = \frac{1}{11}.$$

Here $\alpha = \frac{10}{7}$, $f(t,x(t),{}^{c}D^{\alpha}x(t)) = \frac{1}{10(1+|x(t)|+|{}^{c}D^{\frac{10}{7}}x(t)|)}$, $\lambda = \frac{8}{20}$, $\beta_1 = \frac{6}{14}$, $\beta_2 = \frac{4}{17}$, $\beta_3 = \frac{1}{11}$, $T = 1$, *observe that* $0 < \beta_1, \beta_2 < \frac{10}{7}$. *Hence the hypothesis* (A_2) *holds* *with* $K = L = \frac{1}{10}$ *and we shall check that*

$$\left[\frac{T^{\alpha}}{\Gamma(\alpha+1)} - \frac{T}{\Lambda_1}\left\{\frac{\lambda T^{\alpha-\beta_1+1}}{\Gamma(\alpha-\beta_1+1)} + \frac{(1-\lambda)T^{\alpha-\beta_2+1}}{\Gamma(\alpha-\beta_2+1)}\right\}\right]\frac{K}{(1-L)} < 1$$

$$\Leftrightarrow \qquad \approx 0.8846 < 1.$$

Thus, the theorem 3.1, the fractional boundary value problem (4.6) *and* (4.7) *has a unique solution on* J.

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