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ROUGH IDEALS IN ROUGH NEAR-RINGS

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ABSTRACT. In this paper, we analyzed some concepts regarding rough ring and we introduce concepts of the rough near-ring and its characteristics are demonstrated.

1. INTRODUCTION

The theory of rough set is a conventional mathematical technique to handle imperfect concepts in both application and theoretical part of mathematics. In [4], Z. Pawlak introduced this theory. Furthermore, it has received enormous attention from the scientific communities of mathematics, physics, chemistry, and engineering. The concept of rough set theory is formulated by the pair having universal set and an equivalence relation, where the equivalence relation is indiscernibility relation which denotes the vague knowledge of the universal set. This classifies the subset of the universal set is classified into three categories is known as the boundary region, approximation upper and lower. Many recent research works on rough set theory focuses on the combining or connecting or comparing the concepts of rough set theory with algebra or any algebraic structures. In this paper, we analyze some concepts regarding rough ring and introduce a concept of new algebraic structure called rough near-ring and some of properties regarding ideals are analyzed.

Other valuable references on the topic are [1–3, 5–10].

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2. Basic concepts of Rough sets

Definition 2.1. *An approximation space is known as a pair (X,θ) where the set X is a non-empty and an equivalence relation on set X is θ.*

Definition 2.2. *For a subset B_1 of X in (X,θ), the sets $\overline{B_1}, \underline{B_1}, BN(B_1)$ are usually described as below:*

(1) *An upper approximation of B_1 in (X,θ) is denoted by $\overline{B_1} = \{b \in X/[b]_\theta \cap B_1 \neq \phi\}$ where $[b]_\theta$ signifies the equivalence class determined by b.*
(2) *$\underline{B_1} = \{b \in X/[b]_\theta \subseteq B_1\}$ is known as a lower approximation of B_1 in (X,θ).*
(3) *$BN(B_1) = \overline{B_1} - \underline{B_1}$ is known as a boundary region of B_1 in (X,θ).*

Proposition 2.1. *Let $B_1, B_2 \subset X$, then the following properties are satisfied:*

(1) $\underline{B_1} \subset B_1 \subset \overline{B_1}$;
(2) $\underline{\phi} = \phi = \overline{\phi}, \underline{X} = X = \overline{X}$;
(3) $\underline{B_1 \cap B_2} = \underline{B_1} \cap \underline{B_2}$;
(4) $\overline{B_1 \cap B_2} \subset \overline{B_1} \cap \overline{B_2}$;
(5) $\underline{B_1} \cup \underline{B_2} \subset \underline{B_1 \cup B_2}$;
(6) $\overline{B_1 \cup B_2} = \overline{B_1} \cup \overline{B_2}$;
(7) $B_1 \subset B_2$ *iff* $\underline{B_1} \subset \underline{B_2}$ *and* $\overline{B_1} \subset \overline{B_2}$.

3. Rough Ideals in Near-Ring

Definition 3.1. *Let $+, .$ be two binary operations on X and an approximation space is denoted by (X,θ). A set $N \subset X$ is said to be a Rough Near-ring if it satisfies the following properties:*

I): *1) For all $a_1, a_2 \in N$ implies $a_1 + a_2 \in \overline{N}$*

2) For all $a_1, a_2, a_3 \in N$ implies $(a_1 + a_2) + a_3 = a_1 + (a_2 + a_3)$

3) For all $a_1 \in N$, there exists $e_1 \in \overline{N}$ such that $a_1 + e_1 = a_1 = e_1 + a_1$, where e_1 is called an additive rough near-ring identity.

4) For all $a_1 \in N$, there exists $b_1 \in \overline{N}$ such that $a_1 + b_1 = e_1 = b_1 + a_1$, where b is known an additive rough near-ring inverse.

The above four conditions shows that $(N,+)$ is an additive rough near-ring group.

II): *1) For all* $a_1, a_2 \in N$ *implies* $a_1.a_2 \in \overline{N}$.

2) For all $a_1, a_2, a_3 \in N$ *implies* $(a_1.a_2).a_3 = a_1.(a_2.a_3)$.

The above two conditions shows that $(N, \cdot)$ *signifies a multiplicative rough near-ring semi group.*

III): *1) For all* $a_1, a_2, a_3 \in N$ *implies* $(a_1 + a_2).a_3 = (a_1.a_3) + (a_2.a_3)$.

Definition 3.2. *It is said that a non-empty subset* I *of a rough near-ring* N *is a rough normal subgroup if it meets the following conditions:*

(1) *For all* $a_1, a_2 \in I$ *implies* $a_1 - a_2 \in \overline{I}$.
(2) *For all* $n_1 \in N, i_1 \in I$ *implies* $n_1 + i_1 - n_1 \in \overline{I}$.

Definition 3.3. *It is said that a non-empty subset* I *of a rough near-ring* N *is a rough left ideal in* N *if it meets the following conditions:*

(1) I *is a rough normal subgroup of* $(N, +)$.
(2) *For all* $n_1, n_2 \in N, i_1 \in I$ *implies* $n_1(n_2 + i_1) - n_1n_2 \in \overline{I}$.

Definition 3.4. *It is said that a non-empty subset* I *of a rough near-ring* N *is a rough right ideal in* N *if it meets the following conditions:*

(1) I *is a rough normal subgroup of* $(N, +)$.
(2) *For all* $n_1 \in N, i_1 \in I$ *implies* $i_1.n_1 \in \overline{I}$.

Definition 3.5. *It is said that a non-empty subset* I *of a rough near-ring* N *is a rough ideal in near-ring* N *if it is both rough left and rough right ideal in* N.

4. Homomorphism of Rough Near-Rings

Let (X_1, θ_1) and (X_2, θ_2) be two approximation spaces and $+, .; +', .'$ be binary operations over X_1 and X_2 respectively. Let $N_1 \subseteq X_1$ and $N_2 \subseteq X_2$ be two rough near-rings.

Definition 4.1. *A mapping* $\phi : \overline{N_1} \to \overline{N_2}$ *satisfying*

(1) $\phi(b_1 + b_2) = \phi(b_1) +' \phi(b_2)$.
(2) $\phi(b_1 \cdot b_2) = \phi(b_1).'\phi(b_2)$, *for all* $b_1, b_2 \in \overline{N}_1$ *is known a rough near-ring homomorphism from* N_1 *to* N_2.

Definition 4.2. *A rough near-ring homomorphism* $\phi : N_1 \to N_2$ *is called a rough near-ring epimorphism if the mapping* $\phi : \overline{N_1} \to \overline{N_2}$ *is onto. That is,* $\forall b \in \overline{N_2}, \exists a \in \overline{N_1}$ *such that* $\phi(a) = b$.

Definition 4.3. *A rough near-ring homomorphism $\phi : N_1 \to N_2$ is called a rough near-ring monomorphism if the mapping $\phi : \overline{N_1} \to \overline{N_2}$ is one-one.*

Definition 4.4. *A rough near-ring homomorphism $\phi : N_1 \to N_2$ is called a rough near-ring isomorphism if the mapping $\phi : \overline{N_1} \to \overline{N_2}$ is both one-one and onto.*

Theorem 4.1. *$\phi : \overline{N_1} \to \overline{N_2}$ be a rough near-ring homomorphism. A nonempty subset I_1 of N_1 refers to a left ideal of N_1. Then $\phi(I_1)$ refers to a rough left ideal of N_2 if $\phi(\overline{I}_1) = \overline{\phi(I_1)}$ and $\phi(N_1) = N_2$.*

Proof. For all $a_1, a_2 \in \phi(I_1), \exists\, b_1, b_2 \in I_1$ such that $\phi(b_1) = a_1$ and $\phi(b_2) = a_2$.

(1) We have $\phi(b_1 + b_2) = \phi(b_1) +' \phi(b_2) = a_1 +' a_2$. Since $\phi(b_1 + b_2) \in \phi(\overline{I_1})$, We get $\phi(b_1 + b_2) \in \overline{\phi(I_1)}$. That is, $a_1 +' a_2 \in \overline{\phi(I_1)}$.

(2) Since $0 \in \overline{I_1}$, we get $\phi(0) \in \phi(\overline{I_1}) = \overline{\phi(I_1)}$. Thus, for all $\phi(b_1) \in \phi(I_1)$, there exist $\phi(0) \in \overline{\phi(I_1)}$ such that

$$\phi(b_1) +' \phi(0) = \phi(b_1 + 0) = \phi(b_1) = \phi(0 + b_1) = \phi(0) +' \phi(b_1).$$

(3) Since I is a rough near-ring subgroup, for each $b_1 \in I_1$, additive rough near-ring inverse $-b_1 \in I_1$. Because $-\phi(b_1) = \phi(-b_1) \in \phi(I_1)$, we get $-\phi(b_1) \in \phi(I_1)$. Therefore, $\phi(I_1)$ is a rough subgroup of N_2.

(4) $\phi(n_1) \in N_2$ and $\phi(i_1) \in \phi(I_1)$, then we have

$$\phi(n_1 + i_1 - n_1) = \phi(n_1) +' \phi(i_1) - \phi(n_1) \in \phi(\overline{I_1})$$

This implies, $\phi(n_1) +' \phi(i_1) - \phi(n_1) \in \overline{\phi(I_1)}$. Therefore, $\phi(I_1)$ refers to a rough normal subgroup of N_2.

(5) For all $\phi(n_1), \phi(n_2) \in N_2$ and $\phi(i_1) \in \phi(I_1)$,

$$\phi(n_1 \cdot' (n_2 +' i_1)) - \phi(n_1 \cdot' n_2) = \phi(n_1(n_2 + i_1) - n_1 n_2).$$

We know that I_1 refers to a rough near-ring left ideal. Then we have $n_1(n_2 + i_1) - n_1 n_2 \in \overline{I_1}$. Therefore, $\phi(n_1(n_2 + i_1) - n_1 n_2 \in \phi(\overline{I_1}))$. This leads to, $\phi(n_1).'\phi(n_2 +' i_1) - \phi(n_1) \cdot' \phi(n_2) \in \overline{\phi(I_1)}$. Therefore, $\phi(I_1)$ is a rough left ideal of rough near-ring N_2.

In the same way we can verify the other statement. □

Theorem 4.2. *Let $\phi : \overline{N_1} \to \overline{N_2}$ be a rough near-ring homomorphism and let I_2 be a rough near-ring left ideal of rough near-ring N_2. Then $I_1 = \phi^{-1}(I_2)$ is a rough near-ring left ideal of N_1 if $\phi(\overline{I_1}) = \overline{\phi(I_1)}$ and $\phi(N_1) = N_2$.*

Proof. Since $I_1 = \phi^{-1}(I_2)$, which leads to $\phi(I_1) = I_2$, Then $\overline{I_2} = \overline{\phi(I_1)} = \phi(\overline{I_1})$.

(1) For all $a, b \in I_1$, we have $\phi(a), \phi(b) \in I_2$. Since I_2 is a rough near-ring subgroup, we sustain $\phi(a) +' \phi(b) \in \overline{I_2}$. That is, $\phi(a+b) \in \phi(\overline{I_1})$. Thus, $a + b \in \overline{I_1}$.

(2) $\forall a \in I_1$, we have $\phi(a) \in I_2$, since I_2 is a rough near-ring subgroup, we get $\phi(-a) = -\phi(a) \in I_2$. That is, $\phi(-a) \in \phi(I_1)$. Thus, $-a \in I_1$. Therefore, $I_1 = \phi^{-1}(I_2)$ is a rough near-ring subgroup of N_1.

(3) $\forall \phi(n) \in I_2$ and $\phi(i) \in \phi(I)$, which leads to $\phi(n + i - n) = \phi(n) +' \phi(i) - \phi(n) \in \overline{I_2}$, This implies, $\phi(n + i - n) \in \phi(\overline{I_1})$. That is, $n + i - n \in \overline{I_1}$. Therefore, $I_1 = \phi^{-1}(I_2)$ is a rough normal subgroup of near-ring N_1.

(4) $\forall n, n' \in N$ and $i \in I_1$, which leads to $\phi(n), \phi(n') \in \phi(N_1) = N_2$ and $\phi(i) \in I_2$. Since I_2 be a rough near-ring left ideal of N_2. That is, $\phi(n.'(n' +' i)) - n.'n' \in \phi(\overline{I_1})$. Thus, $(n.'(n' +' i)) - n.'n' \in \overline{I_1}$. Therefore, $I_1 = \phi^{-1}(I_2)$ is a rough near-ring left ideal of N_1.

In the similar procedure we can verify the other statement. □

5. Anti-Homomorphism of Rough Near-Rings

Definition 5.1. *A mapping $\phi : \overline{N_1} \to \overline{N_2}$ satisfying*

I: $\phi(b_1 + b_2) = \phi(b_2) +' \phi(b_1)$

II: $\phi(b_1.b_2) = \phi(b_2).'\phi(b_1)$ *for all $b_1, b_2 \in \overline{N}_1$, is called a rough near-ring anti-homomorphism from N_1 to N_2.*

Theorem 5.1. *Let $\phi : \overline{N_1} \to \overline{N_2}$ be a rough near-ring anti-homomorphism. Let I_1 be a rough right ideal of a rough near-ring N_1. Then $\phi(I_1)$ refers to a rough right ideal of N_2 if $\phi(\overline{I}_1) = \overline{\phi(I_1)}$ and $\phi(N_1) = N_2$.*

Proof. For all $a_1, a_2 \in \phi(I_1), \exists b_1, b_2 \in I_1$ such that $\phi(b_1) = a_1$ and $\phi(b_2) = a_2$.

(1) $\phi(b_2 + b_1) = \phi(b_1) +' \phi(b_2) = a_1 +' a_2$
Since $\phi(b_2+b_1) \in \phi(\overline{I}_1)$, We get $\phi(b_2+b_1) \in \overline{\phi(I_1)}$. That is, $a_1+'a_2 \in \overline{\phi(I_1)}$.

(2) Since $0 \in \overline{I_1}$, we get $\phi(0) \in \phi(\overline{I}_1) = \overline{\phi(I_1)}$.
Thus, for all $\phi(b_1) \in \phi(I_1)$, there exist $\phi(0) \in \overline{\phi(I_1)}$ such that

$$\phi(b_1) +' \phi(0) = \phi(0 + b_1) = \phi(b_1) = \phi(b_1 + 0) = \phi(0) +' \phi(b_1).$$

(3) Since I is a rough near-ring subgroup, for each $b_1 \in I_1$, additive rough near-ring inverse $-b_1 \in I_1$. Because $-\phi(b_1) = \phi(-b_1) \in \phi(I_1)$, we get $-\phi(b_1) \in \phi(I_1)$. Therefore, $\phi(I_1)$ is a rough subgroup of N_2.

(4) For all $\phi(n_1) \in N_2$ and $\phi(i_1) \in \phi(I_1)$, then we have

$$\begin{aligned}\phi(n_1 + i_1 - n_1) &= \phi(-n_1) +^{'} \phi(n_1 + i_1)\\ &= -\phi(n_1) +^{'} \phi(i_1) +^{'} \phi(n_1) \in \phi(\bar{I}_1)\end{aligned}$$

This implies, $\phi(n_1 + i_1 - n_1) \in \overline{\phi(I_1)}$. Therefore, $\phi(I_1)$ refers to a rough normal subgroup of N_2.

(5) For all $n_1 \in N_1$ and $i_1 \in I_1$, we have $\phi(n_1) \in N_2$ and

$$\phi(i_1) \in I_1, \phi(n_1).^{'}\phi(i_1) = \phi(i_1.n_1).$$

We know that I is right ideal with rough set, we have $i_1.n_1 \in \overline{I_1}$. Therefore $\phi(i_1.n_1) \in \phi(\overline{I}_1)$. Thus $\phi(n_1).^{'}\phi(i_1) \in \overline{\phi(I_1)}$. Therefore, $\phi(I_1)$ is a right ideal of rough near-ring N_2.

In the similar procedure, we can verify the other statement. □

Theorem 5.2. *Let $\phi : \overline{N_1} \to \overline{N_2}$ be a rough near-ring anti-homomorphism and let I_2 be a rough right ideal of rough near-ring N_2. Then $I_1 = \phi^{-1}(I_2)$ is a rough right ideal of near-ring N_1 if $\phi(\overline{I}_1) = \overline{\phi(I_1)}$ and $\phi(N_1) = N_2$.*

Proof. Since $I_1 = \phi^{-1}(I_2)$, one obtain $\phi(I_1) = I_2$, and so $\overline{I_2} = \overline{\phi(I_1)} = \phi(\overline{I_1})$.

(1) For all $a, b \in I_1$, we have $\phi(a), \phi(b) \in I_2$.
Since I_2 is a rough near-ring subgroup, We get $\phi(b) +^{'} \phi(a) \in \overline{I_2}$.
That is, $\phi(a + b) \in \phi(\overline{I_1})$. Thus, $a + b \in \overline{I_1}$.

(2) $\forall a \in I_1$, we have $\phi(a) \in I_2$,
since I_2 is a rough near-ring subgroup, we get $\phi(-a) = -\phi(a) \in I_2$.
That is, $\phi(-a) \in \phi(I_1)$. Thus, $-a \in I_1$.
Therefore, $I_1 = \phi^{-1}(I_2)$ is a rough near-ring subgroup of N_1.

(3) $\forall \phi(n) \in I_2$ and $\phi(i) \in \phi(I)$, we have $\phi(n + i - n) = \phi(-n) +^{'} \phi(n + i) = \phi(-n) +^{'} \phi(i) +^{'} \phi(n) = -\phi(n) +^{'} \phi(i) +^{'} \phi(n) \in \overline{I_2}$.
This implies, $\phi(n + i - n) \in \phi(\overline{I_1})$. That is, $n + i - n \in \overline{I_1}$.
Therefore, $I_1 = \phi^{-1}(I_2)$ is a rough normal subgroup of near-ring N_1.

(4) $\forall n, \in N$ and $a \in I_1$, we have $\phi(n) \in \phi(N_1) = N_2$ and $\phi(a) \in I_2$. Since I_2 be a rough right ideal of near-ring of N_2, we have $\phi(n).^{'}\phi(i) \in \overline{I_2}$. Thus,

$\phi(i \cdot n) \in \phi(\overline{I_1})$. That is, $i, n \in \overline{I_1}$. Therefore, $I_1 = \phi^{-1}(I_2)$ is a rough right ideal of near-ring N_1.

In the similar procedure we can verify the other statement. □

6. CONCLUSION

This study presents the concept regarding rough near-ring and we analyzed the concepts such as rough near-ring homomorphism and rough near-ring anti-homomorphism with respect to ideals. Likewise rough sets can be applied or extended to any algebraic structures and its concepts or properties can be examined.

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