Advances in Mathematics: Scientific Journal **9** (2020), no.5, 2573–2582
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.5.20

ON ir-CLOSED SETS IN TOPOLOGICAL SPACES

K. BALA DEEPA ARASI[1], J. ANUF SARUMATHI[2], V. MAHESWARI[3], AND Y. KAYATHIRI[4]

ABSTRACT. The aim of this paper is to introduce the concept of ir- closed sets in topological spaces. A subset A of a topological space X is called an ir- closed set if $rcl(A) \subseteq U$ whenever $A \subseteq U$ and U is i- open in X. Also we investigate their properties and study the relationship with other existing generalized closed sets.

1. INTRODUCTION

In 1970, Levine [3] introduced the concept of generalized closed sets as a weaker form of closed sets in topological spaces. Mohammed and Askander in 2012 [1] introduced the concept of i- open sets. Regular open sets have been introduced and investigated by Stone [4].

In this paper, we introduce ir- closed set in topological space and investigate the relationship with other known classes of closed sets. Also we discuss their properties and characterize the ir- closed set.

2. PRELIMINARIES

In this section, we recollect some basic preliminaries for some of the relevant open and closed sets in topological spaces.

Definition 2.1. *A subset A of a topological space X is called a*

[1]*corresponding author*

2010 *Mathematics Subject Classification.* 54A05, 54A10.
Key words and phrases. ir- closed, i- open, r- closure.

(i) *regular closed (briefly r- closed) if* $A = cl(int(A))$
(ii) *pre-closed if* $cl(int(A)) \subseteq A$
(iii) *semi-open if* $A \subseteq cl(int(A))$
(iv) α- *open if* $A \subseteq int(cl(int(A)))$
(v) *semi-preclosed if* $int(cl(int(A))) \subseteq A$.

Proposition 2.1. *Every regular open set is open in* X.

Definition 2.2. *A subset* A *of a topological space* X *is called an* i- *open set if there exists an open set* $G \in \tau(X)$ *such that* $G \neq \emptyset, X$ *and* $A \subseteq cl(A \cap G)$. *The complement of an* i- *open set is an* i- *closed set.*

Proposition 2.2. *Every open set in a topological space is* i- *open, but the converse is not true [1].*

Proposition 2.3. *Every semi-open set is* i- *open [1].*

We now recall the definitions of some of the generalized closed sets in a topological space.

Definition 2.3. *A subset* A *of a topological space* X *is called a*

(1) *Regular generalised closed set (briefly* rg- *closed) if* $cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is regular open in* (X, τ).
(2) *Generalised preregular closed set (briefly* gpr- *closed) if* $pcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is regular open in* (X, τ).
(3) *Regular weakly generalised closed set (briefly* rwg- *closed) if* $cl(int(A)) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is regular open in* (X, τ).
(4) *Weakly generalised closed set (briefly* wg- *closed) if* $cl(int(A)) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* (X, τ).
(5) *Generalised pre closed set (briefly* gp- *closed) if* $pcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* (X, τ).
(6) *Generalised closed set (briefly* g- *closed set) if* $cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* X. *The complement of a* g- *closed set is* g- *open set.*
(7) *Semi-generalised closed set (briefly* sg- *closed set) if* $scl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is semi-open in* X.
(8) *Generalised semi-closed set (briefly* gs- *closed set) if* $cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* X.
(9) *Generalised semi-preclosed set (briefly* gsp- *closed set) if* $spcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* X.

(10) *$\hat{g}$- closed set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X.*
(11) *α- generalised closed set (briefly αg- closed set) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.*
(12) *Generalised α- closed set (briefly $g\alpha$- closed set) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α- open in X.*
(13) *$sb\hat{g}$- closed set [2] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is $b\hat{g}$- open in X.*
(14) *g^*- closed set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g- open in X.*
(15) *b^*- closed set if $int(cl(A)) \subseteq U$ whenever $A \subseteq U$ and U is b- open in X.*
(16) *$r\hat{g}$- closed set if $rcl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$- open in (X, τ).*

3. ir- CLOSED SETS IN TOPOLOGICAL SPACES

We introduce and study the notion of ir- closed sets and obtain some of its basic properties.

Definition 3.1. *A subset A of a topological space X is called an ir- closed set if $rcl(A) \subseteq U$ whenever $A \subseteq U$ and U is i- open in X. The complement of an ir- closed set is ir- open set.*

Example 1. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$.*
Then r- closed $= \{\emptyset, X, \{a, b\}, \{c\}\}$; ir- closed set$= \{\emptyset, X, \{c\}, \{a, b\}, \{a, c\}\}$.
Here $A = \{a, c\}$ is ir- closed set but not r- closed.

Proposition 3.1. *Every regular open is i- open.*

Proof. Since every regular open is open and every open is i- open implies every regular open is i- open. □

Theorem 3.1. *Every ir- closed set is rg- closed set.*

Proof. Let A be an ir- closed set in (X, τ). Let $A \subseteq U$ and U is regular open. Since every regular open is i- open, U is i- open. Since every regular closed set is closed, $cl(A) \subseteq rcl(A) \subseteq U$ where U is regular open.
Therefore A is rg- closed set in (X, τ).
The converse need not be true. □

Example 2. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a, b\}\}$.*
Then rg- closed set $=\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

ir- closed set $=\{\emptyset, X, \{c\}, \{a,c\}, \{b,c\}\}$.
Here the sets $\{a\}, \{b\}, \{a,b\}$ are rg- closed set but not ir- closed set.

Theorem 3.2. *Every ir- closed set is gpr- closed set.*

Proof. Let A be an ir- closed set in (X, τ). Let $A \subseteq U$ and U is regular open. Since every regular open is i- open, U is i- open. Since every preclosed set is closed and regular closed set is closed, $pcl(A) \subseteq cl(A) \subseteq rcl(A) \subseteq U$ where U is regular open.
Therefore A is gpr- closed set in (X, τ).
The converse need not be true. □

Example 3. *Let $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{b\}, \{c\}, \{a,b\}, \{b,c\}\}$.*
Then gpr- closed set $= \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$.
ir- closed set $= \{\emptyset, X, \{c\}, \{a,b\}, \{a,c\}\}$.
Here the sets $\{a\}, \{b\}, \{b,c\}$ are gpr- closed set but not ir- closed set.

Theorem 3.3. *Every ir- closed set is rwg- closed set.*

Proof. Let A be an ir- closed set in (X, τ). Let $A \subseteq U$ and U is regular open. Since every regular open is i- open, U is i- open, $cl(int(A)) \subseteq cl(A) \subseteq rcl(A) \subseteq U$ where U is regular open. Therefore A is rwg- closed set in (X, τ).
The converse need not be true. □

Example 4. *Let $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$.*
Then rwg- closed set $=\{\emptyset, X, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$.
ir- closed set $= \{\emptyset, X, \{a,c\}, \{b,c\}\}$.
Here the sets $\{c\}, \{a,b\}$ are rwg- closed set but not ir- closed set.

Theorem 3.4. *Every ir- closed set is wg- closed set.*

Proof. Let A be an ir- closed set in (X, τ). Let $A \subseteq U$ and U is open.
Since every open is i- open, U is i- open, $cl(int(A)) \subseteq cl(A) \subseteq rcl(A) \subseteq U$ where U is open. Therefore A is wg- closed set in (X, τ).
The converse need not be true. □

Example 5. *Let $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{c\}, \{a,c\}, \{b,c\}\}$.*
Then wg- closed set $= \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$;
ir- closed set $= \{\emptyset, X, \{a,b\}\}$.
Here the sets $\{a\}, \{b\}$ are wg- closed set but not ir- closed set.

Theorem 3.5. *Every ir- closed set is g- closed set.*

Proof. Let A be an ir- closed set in (X,τ). Let $A \subseteq U$ and U is open.
Since every open is i- open, U is i- open, $cl(A) \subseteq rcl(A) \subseteq U$ where U is open.
Therefore A is g- closed set in (X,τ).
The converse need not be true. □

Example 6. *Let $X=\{a,b,c\}$ and $\tau=\{\emptyset,X,\{a\},\{b\},\{a,b\}\}$.*
Then g- closed set $=\{\emptyset,X,\{c\},\{a,c\},\{b,c\}\}$;
ir- closed set $=\{\emptyset,X,\{a,c\},\{b,c\}\}$.
Here the set $\{c\}$ is g- closed set but not ir- closed set.

Theorem 3.6. *Every ir- closed set is gs- closed set.*

Proof. Let A be an ir- closed set in (X,τ). Let $A \subseteq U$ and U is open.
Since every open is i- open, U is i- open, $scl(A) \subseteq cl(A) \subseteq rcl(A) \subseteq U$ where U is regular open. Therefore A is gs- closed set in (X,τ).
The converse need not be true. □

Example 7. *Let $X=\{a,b,c\}$ and $\tau=\{\emptyset,X,\{c\}\}$.*
Then gs- closed set $=\{\emptyset,X,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\}\}$;
ir- closed set $=\{\emptyset,X,\{a,b\}\}$.
Here the sets $\{a\},\{b\},\{a,c\},\{b,c\}$ are gs- closed set but not ir- closed set.

Theorem 3.7. *Every ir- closed set is αg- closed set.*

Proof. Let A be an ir- closed set in (X,τ). Let $A \subseteq U$ and U is open.
Since every open is i- open, U is i- open. Since every closed is α- closed and every regular closed is closed, $\alpha cl(A) \subseteq cl(A) \subseteq rcl(A) \subseteq U$ where U is open. Therefore A is αg- closed set in (X,τ).
The converse need not be true. □

Example 8. *Let $X=\{a,b,c\}$ and $\tau=\{\emptyset,X,\{c\},\{b,c\}\}$.*
Then αg- closed set $=\{\emptyset,X,\{a\},\{b\},\{a,b\},\{a,c\}\}$;
ir- closed set $=\{\emptyset,X\}$.
Here the sets $\{a\},\{b\},\{a,b\},\{a,c\}$ are αg- closed set but not ir- closed set.

Theorem 3.8. *Every ir- closed set is gp- closed set.*

Proof. Let A be an ir- closed set in (X,τ). Let $A \subseteq U$ and U is open.

Since every open is i- open, U is i- open, $pcl(A) \subseteq cl(A) \subseteq rcl(A) \subseteq U$ where U is open. Therefore A is gp- closed set in (X, τ).

The converse need not be true. □

Example 9. *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, X, \{a, b\}\}$.

Then gp*- closed set* $= \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$;

ir*- closed set* $= \{\emptyset, X, \{c\}, \{a, c\}, \{b, c\}\}$.

Here the sets $\{a\}, \{b\}$ *are* gp*- closed set but not* ir*- closed set.*

Theorem 3.9. *Every* ir*- closed set is* sg*- closed set.*

Proof. Let A be an ir- closed set in (X, τ). Let $A \subseteq U$ and U is semi-open.
Since every semi-open is i- open, U is i- open. Since every closed is semi-closed and every regular closed is closed, $scl(A) \subseteq cl(A) \subseteq rcl(A) \subseteq U$ where U is semi-open. Therefore A is sg- closed set in (X, τ).

The converse need not be true. □

Example 10. *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$.

Then sg*- closed set* $= \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$;

ir*- closed set* $= \{\emptyset, X, \{a, c\}, \{b, c\}\}$.

Here the sets $\{a\}, \{b\}, \{c\}$ *are* sg*- closed set but not* ir*- closed set.*

Theorem 3.10. *Every* ir*- closed set is ĝ- closed set.*

Proof. Let A be an ir- closed set in (X, τ). Let $A \subseteq U$ and U is semi-open.
Since every semi-open is i- open, U is i- open, $cl(A) \subseteq rcl(A) \subseteq U$ where U is semi-open. Therefore A is ĝ- closed set in (X, τ).

The converse need not be true. □

Example 11. *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$.

Then ĝ- closed set $= \{\emptyset, X, \{c\}, \{a, c\}, \{b, c\}\}$;

ir*- closed set* $= \{\emptyset, X, \{a, c\}, \{b, c\}\}$.

Here the set $\{c\}$ *is ĝ- closed set but not* ir*- closed set.*

Theorem 3.11. *Every* ir*- closed set is* gsp*- closed set.*

Proof. Let A be an ir- closed set in (X, τ). Let $A \subseteq U$ and U is open.
Since every open is i- open, U is i- open. Since every closed is semi pre-closed and every regular closed is closed, $spcl(A) \subseteq cl(A) \subseteq rcl(A) \subseteq U$ where U is open. Therefore A is gsp- closed set in (X, τ).

The converse need not be true. □

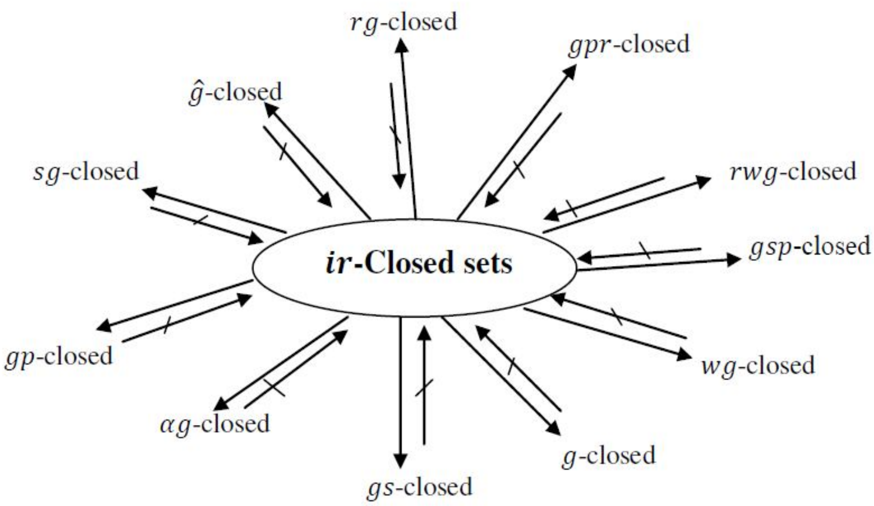
Example 12. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{c\}\}$.*

Then gsp- closed set $= \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$;

ir- closed set $= \{\emptyset, X, \{a, b\}\}$.

Here the sets $\{a\}, \{b\}, \{a, c\}, \{b, c\}$ are gsp- closed set but not ir- closed set.

Remark 3.1. *The following diagram shows the relationship of ir- closed sets with other known existing sets.*



Theorem 3.12. *Union of two ir- closed set is ir- closed set.*

Proof. Let A and B be two ir- closed sets.

Let G be any i- open set in (X, τ), such that $A \cup B \subseteq G$.

Then $A \subseteq G$ and $B \subseteq G$. Since A and B are ir- closed set, $rcl(A) \subseteq G$ and $rcl(B) \subseteq G$.

Therefore $rcl(A) \cup rcl(B) = rcl(A \cup B) \subseteq G$.

Hence $A \cup B$ is ir- closed set. □

Result 1. *The intersection of ir- closed sets need not be ir- closed set.*

Example 13. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{c\}, \{a, b\}\}$.*

Then ir- closed set$= \{\emptyset, X, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

Let $A = \{a, b\}$ and $B = \{a, c\}$ where $A \cap B = \{a\}$ which is not an ir- closed set.

Remark 3.2. *g^*- closed set and ir- closed sets are independent of each other.*

Example 14.

(i) *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, X, \{c\}, \{a, b\}\}$.
Then g^**- closed set* $= \{\emptyset, X, \{c\}, \{a, b\}\}$.
ir*- closed set* $= \{\emptyset, X, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.
The sets $\{a, c\}, \{b, c\}$ *are* ir*- closed set but not* g^**- closed set.*

(ii) *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, X, \{c\}, \{b, c\}\}$.
Then g^**- closed set* $= \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$.
ir*- closed set* $= \{\emptyset, X\}$. *The sets* $\{a\}, \{a, b\}, \{a, c\}$ *are* g^**- closed set but not* ir*- closed set.*

Remark 3.3. $g\alpha$*- closed set and* ir*- closed sets are independent of each other.*

Example 15.

(i) *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, X, \{c\}\}$.
Then $g\alpha$*- closed set*$= \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$.
ir*- closed set* $= \{\emptyset, X, \{a, b\}\}$.
The sets $\{a\}, \{b\}$ *are* $g\alpha$*- closed set but not* ir*- closed.*

(ii) *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, X, \{a, b\}\}$.
Then $g\alpha$*- closed set*$= \{\emptyset, X, \{c\}\}$.
ir*- closed set* $= \{\emptyset, X, \{c\}, \{a, c\}, \{b, c\}\}$.
The sets $\{a, c\}, \{b, c\}$ *are* ir*- closed set but not* $g\alpha$*- closed set.*

Remark 3.4. α*- closed set and* ir*- closed sets are independent of each other.*

Example 16.

(i) *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, X, \{c\}\}$.
Then α*- closed set*$= \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$.
ir*- closed set* $= \{\emptyset, X, \{a, b\}\}$.
The sets $\{a\}, \{b\}$ *are* α*- closed set but not* ir*- closed.*

(ii) *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, X, \{a, b\}\}$.
Then α*- closed set*$= \{\emptyset, X, \{c\}\}$.
ir*- closed set* $= \{\emptyset, X, \{c\}, \{a, c\}, \{b, c\}\}$.
The sets $\{a, c\}, \{b, c\}$ *are* ir*- closed set but not* α*- closed set.*

Remark 3.5. b^**- closed set and* ir*- closed sets are independent of each other.*

Example 17.

(i) *Let*$X = \{a, b, c\}$ *and* $\tau = \{\emptyset, X, \{c\}\}$.
Then b^**- closed set*$= \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$.
ir*- closed set* $= \{\emptyset, X, \{a, b\}\}$.
The sets $\{a\}, \{b\}$ *are* b^**- closed set but not* ir*- closed set.*

(ii) *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, X, \{a, b\}\}$.
Then b^**- closed set*$= \{\emptyset, X, \{c\}\}$.
ir*- closed set* $= \{\emptyset, X, \{c\}, \{a, c\}, \{b, c\}\}$.
The sets $\{a, c\}, \{b, c\}$ *are* ir*- closed set but not* b^**- closed set.*

Remark 3.6. *$r\hat{g}$- closed set and ir- closed sets are independent of each other.*

Example 18.

(i) *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, X, \{c\}, \{b, c\}\}$.
Then $r\hat{g}$*- closed set*$= \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$*;*
ir*- closed set* $= \{\emptyset, X\}$.
The sets $\{a\}, \{a, b\}, \{a, c\}$ *are* $r\hat{g}$*- closed set but not* ir*- closed set.*

(ii) *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, X, \{c\}, \{a, b\}\}$.
Then r&g*- closed set*$= \{\emptyset, X, \{c\}, \{a, b\}\}$.
ir*- closed set* $= \{\emptyset, X, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.
The sets $\{a, c\}, \{b, c\}$ *are* ir*- closed set but not* $r\hat{g}$*- closed set.*

Remark 3.7. *$sb\hat{g}$- closed set and ir- closed sets are independent of each other.*

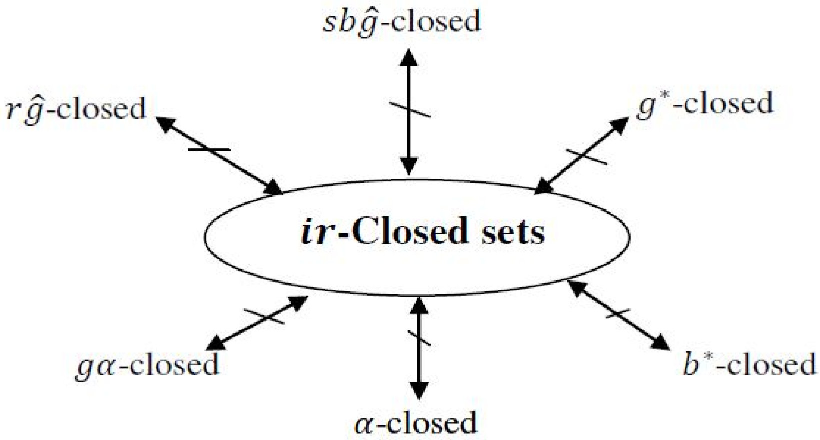
Example 19.

(i) *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, X, \{c\}\}$.
Then $sb\hat{g}$*- closed set*$= \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$;
ir*- closed set* $= \{\emptyset, X, \{a, b\}\}$.
The sets $\{a\}, \{b\}$ *are* $sb\hat{g}$*- closed set but not* ir*- closed*

(ii) *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, X, \{a, b\}\}$.
Then $sb\hat{g}$*- closed set*$= \{\emptyset, X, \{c\}\}$.
ir*- closed set* $= \{\emptyset, X, \{c\}, \{a, c\}, \{b, c\}\}$.
The sets $\{a, c\}, \{b, c\}$ *are* ir*- closed set but not* $sb\hat{g}$*- closed set.*

Remark 3.8. *The following diagram shows the independence of ir- closed sets with other known existing sets.*

Theorem 3.13. *If A is i- open and ir- closed set in (X, τ) then A is semi-closed.*



Proof. Since A is i- open and ir- closed we have $int(cl(A)) \subseteq rcl(A) \subseteq A$. Therefore A is semi-closed. □

REFERENCES

[1] A. A. MOHAMMED, S. W. ASKANDAR: *On i-open sets*, UAE Math Day Conference, American Univ. of Sharjah, April 14, 2012.

[2] K. B. ARASI, S. N. KRISHNAN: *On sbĝ-closed sets in topological spaces*, International Journal of Mathematical Archive, **6**(10) (2015), 115–121.

[3] N. LEVINE: *Generalised closed sets in topology*, RendCire. Math. Palermo., **19**(2) (1970), 89–96.

[4] M. STONE: *Application of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc., **41**(1937), 375–481.

[1,2,3,4]PG AND RESEARCH DEPARTMENT OF MATHEMATICS, A.P.C MAHALAXMI COLLEGE FOR WOMEN, THOOTHUKUDI.

E-mail address: [1]`baladeepa85@gmail.com`

E-mail address: [2]`anufsarumathi96@gmail.com`

E-mail address: [3]`mahiraj2005@gmail.com`

E-mail address: [4]`kayathiri906@gmail.com`