ISOMORPHISMS IN NORMAL CATEGORIES OF UNIT REGULAR SEMIGROUPS

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ABSTRACT. A semigroup S is said to be unit regular if $1 \in S$ and for each element $s \in S$ there exists an element u in the group of units G of S such that $s = sus$. The concept of normal category was introduced by K.S.S. Namboori-pad in the context of describing cross-connections for regular semigroups. The normal category associated with a regular semigroup is the category $\mathcal{L}(S)$ of principal left ideals with right translations as morphisms.In the case of unit regular semigroups the isomorphisms in the normal category are determined by the group of units of the semigroup. We characterise these isomorphisms and give simplified descriptions for the isomorphisms in the normal category arising from unit regular semigroups.

1. INTRODUCTION

The concept of unit regularity has its roots in ring theory and unit regular rings were introduced by Ehrlich [1]. Unit regularity for semigroups is adapted from that for rings.The concept of normal category was introduced by K.S.S. Nambooripad in the context of describing structure of regular semigroups using cross-connections [2]. Here we consider isomorphisms in normal categories associated with unit regular semigroups and give simplified descriptions for the isomorphisms in normal categories.

2. NORMAL CATEGORIES AND UNIT REGULAR SEMIGROUPS

Definition 2.1. *[5] A semigroup S is called unit regular if $1 \in S$ and for each element $s \in S$ there exists an element u in the group of units G of S such that $s = sus$.*

Hence both su and us are idempotents and we can write $s = u^{-1}us$ or $s = suu^{-1}$. Thus every element of S is a product of a group element and an idempotent.

Definition 2.2. *([3], [4]) The normal category $\mathcal{L}(S)$ of principal left ideals of a regular semigroup S is described as follows. The vertex set is the set of all principal left ideals of S and is given by*

$$v\mathcal{L}(S) = \{Se : e \in E(S)\},$$

where $E(S)$ is the set of all idempotents in S. A morphism $\rho : Se \to Sf$ is a right translation $x \mapsto xu$ for some $u \in eSf$ and is denoted by $\rho(e, u, f) : Se \to Sf$.

The morphisms in $\mathcal{L}(S)$ are characterized as follows:

Lemma 2.1. *Let $\rho : Se \to Sf$ be a morphism in $\mathcal{L}(S)$. Then $\rho = \rho(e, u, f)$ where $u = \rho(e) \in eSf$. Further every morphism from Se to Sf arises as $\rho(e, u, f)$ for some $u \in eSf$.*

Proposition 2.1. *[2] Se and Sf are isomorphic in $\mathcal{L}(S)$ if and only if $e\mathcal{D}f$. In this case, there is a bijection between the set of all isomorphisms of Se onto Sf and the $\mathcal{H}$-class $\mathcal{R}_e \cap \mathcal{L}_f$. Here $\mathcal{L}$ and $\mathcal{R}$ are Greens relations and $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$.*

Proposition 2.2. *A morphism $\rho(e, u, f) : Se \to Sf$ is an isomorphism if and only if $e\mathcal{R}u\mathcal{L}f$.*

3. ISOMORPHISMS IN NORMAL CATEGORIES OF UNIT REGULAR SEMIGROUPS

Here we consider unit regular semigroups with the additional property that any two $\mathcal{D}-$related idempotents are conjugates. These are called strongly unit regular semigroups. If S is a strongly unit regular semigroup then for $e, f \in E(S)$ and if $e\mathcal{D}f$ there is a unit $u \in S$ such that $f = u^{-1}eu$. Further in this case for $x, y \in S$, $x\mathcal{R}y$ if and only if $xu = y$ for some $u \in G$, $x\mathcal{L}y$ if and only if $ux = y$

for some $u \in G$ and $x\mathcal{D}y$ if and only if $y = uxu^{-1}$ for some $u \in G$, where G is the group of units of S.

Theorem 3.1. *Let S be a unit regular semigroup with group of units G. Let $\rho(e, x, f) : Se \to Sf$ be an isomorphism. Then $\rho(e, x, f) = \rho(e, eu, f)$ for some $u \in G$.*

Proof. Since $\rho(e, x, f) : Se \to Sf$ is an isomorphism, by Proposition 2.2, we have $e\mathcal{R}x\mathcal{L}f$. By the property of strongly unit regular semigroups, $e\mathcal{R}x$ gives $x = eu$ for some $u \in G$. Hence $\rho(e, x, f) = \rho(e, eu, f)$ for some $u \in G$. $\qquad\square$

Theorem 3.2. *For a unit regular semigroup S, $\rho(e, eu, f) : Se \to Sf$ is an isomorphism if and only if $Sf = S(u^{-1}eu)$. Further in this case the inverse of $\rho(e, eu, f)$ is $\rho(f, fu^{-1}, e)$.*

Proof. Let $\rho(e, eu, f) : Se \to Sf$ be an isomorphism. Then by Proposition 2.2, we have $e\mathcal{R}eu\mathcal{L}f$. Thus $eu\mathcal{L}f$. Also from the defining property of Green's $\mathcal{L}-$relation, $u^{-1}eu\mathcal{L}eu$. Hence $u^{-1}eu\mathcal{L}f$. Therefore, $Sf = S(u^{-1}eu)$.

Conversely, suppose $Sf = S(u^{-1}eu)$. Then we have $f\mathcal{L}u^{-1}eu$. By the property of strongly unit regular semigroups we get an element $v \in G$, where G is the group of units of S such that $vf = u^{-1}eu$ and thus we have $f = (v^{-1}u^{-1})eu$ and $eu = (uv)f$. Hence $eu\mathcal{L}f$. Also $e = euu^{-1}$ and so $e\mathcal{R}eu$. Thus $e\mathcal{R}eu\mathcal{L}f$. Hence by Proposition 2.2 we get $\rho(e, eu, f) : Se \to Sf$ is an isomorphism.

To see that the map $\rho(f, fu^{-1}, e)$ is a morphism from Sf to Se we prove using Lemma 2.1 that $fu^{-1} \in fSe$. Since $\rho(e, eu, f)$ is an isomorphism, we have $f\mathcal{L}eu$. So $fu^{-1}\mathcal{L}euu^{-1} = e$ and so $fu^{-1}e = fu^{-1}$. Thus $fu^{-1} \in fSe$.

Now for any $x \in Se$,

$$
\begin{aligned}
x\rho(e, eu, f)\rho(f, fu^{-1}, e) &= (xeu)fu^{-1} \\
&= x(euf)u^{-1} \\
&= xeuu^{-1}, \text{ since } eu\mathcal{L}f \\
&= xe \\
&= x, \text{ since } x \in Se
\end{aligned}
$$

So, $\rho(e, eu, f)\rho(f, fu^{-1}, e) = 1_{Se}$. Similarly, for $x \in Sf$,

$$x\rho(f, fu^{-1}, e)\rho(e, eu, f) = xfu^{-1}eu$$
$$= xf, \text{since } u^{-1}eu = f$$
$$= x$$

So, $\rho(f, fu^{-1}, e)\rho(e, eu, f) = 1_{Sf}$. Hence inverse of $\rho(e, eu, f)$ is $\rho(f, fu^{-1}, e)$. $\square$

Corollary 3.1. *Let Se, Sf be isomorphic objects in $\mathcal{L}(S)$. Then the set of all isomorphisms from Se to Sf is $G(Se, Sf) = \{\rho(e, eu, u^{-1}eu) : u \in G\}$ where $u^{-1}eu\mathcal{L}f$. Further $\rho(e, eu, f) = \rho(e, ev, f)$ if and only if $G(e)u = G(e)v$, where $G(e) = \{u \in G : eu = e\}$.*

Proof. By Theorem 3.1 and Theorem 3.2, we get the set of all isomorphisms from Se to Sf is $G(Se, Sf) = \{\rho(e, eu, u^{-1}eu) : u \in G\}$ where $u^{-1}eu\mathcal{L}f$. Further, suppose $G(e)u = G(e)v$. Then for $u_1 \in G(e)$ there exists $v_1 \in G(e)$ such that $u_1u = v_1v$. Now for $x \in Se$, consider,

$$x\rho(e, eu, f) = xeu$$
$$= xeu_1u, \text{since } eu_1 = e$$
$$= xev_1v$$
$$= xev, \text{since } ev_1 = e$$
$$= x\rho(e, ev, f)$$

Hence $\rho(e, eu, f) = \rho(e, ev, f)$.
Now suppose $\rho(e, eu, f) = \rho(e, ev, f)$. Then $eu = ev$. Let $u' \in G(e)u$. Hence $u' = u_1u$ for some $u_1 \in G(e)$. Now $u_1u = (u_1uv^{-1})v$. Also $eu_1uv^{-1} = euv^{-1} = evv^{-1} = e \implies u_1uv^{-1} \in G(e) \implies u' = u_1u = (u_1uv^{-1})v \in G(e)v$. Thus $G(e)u \subseteq G(e)v$. Similarly, $G(e)v \subseteq G(e)u$. Hence $G(e)u = G(e)v$. $\square$

4. CONCLUSION

We considered normal category associated with strongly unit regular semigroups and we see that each isomorphism from $Se \to Sf$ is determined by a group coset $G(e)u$, where $G(e)$ is a subgroup of the group G of units of the semigroup S.

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