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ON PAIRED-DOUBLE DOMINATION NUMBER OF GRAPHS

M. N. SREE VALLI[1] AND V. ANUSUYA

ABSTRACT. A Paired-dominating set of a graph G is a dominating set of vertices whose induced subgraph has a perfect matching and a double dominating set is a dominating set that dominates every vertex of G at least twice. A Paired-double dominating set of a graph G is a double dominating set of vertices whose induced subgraph has a perfect matching. We show that for path and cycle, the double domination number less than or equal to the paired double domination number. Then we characterize the path and cycle having equal paired double domination numbers and double domination number.

1. INTRODUCTION

Let $G = (V, E)$ be a graph with vertex set V and edge set E. We begin with some terminology. For a vertex v of a graph G, *the open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V/ \ uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$.

A subset $S \subseteq V$ is a *dominating set* of G, if for every vertex $v \in V$, $|N[v] \cap S| \geq 1$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A subset S of V is a *double dominating set* of G if for every vertex $v \in V$, $|N[v] \cap S| \geq 2$, that is v is in S and has at least one neighbor in S or v is in $V - S$ and has at least two neighbors in S [2]. A set S is called *paired−dominating set* if it dominates V and the induced subgraph $\langle S \rangle$ contains at least one perfect

[1]*corresponding author*

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matching. A paired-dominating set S with $matching\ M$ is a dominating set $S = \{v_1, v_2, \ldots, v_{2t-1}, v_{2t}\}$ with independent edge set $M = \{e_1, e_2, \ldots, e_t\}$ where each edge e_i joins two elements of S, that is M is a perfect matching in the induced subgraph $\langle S \rangle$. If $v_j v_k = e_i \in M$, we say that v_j and v_k are paired in S [5]. The $double\ domination\ number\ \gamma_{dd}(G)$ is the minimum cardinality of a double dominating set of G, and the $paired-domination\ number\ \gamma_{pr}(G)$ is the minimum cardinality of a paired-dominating set of G. See also [1] and [4].

A paired (respectively, double) dominating set of minimum cardinality is called a $\gamma_{pr}(G)$ set (respectively $\gamma_{dd}(G)$ set). Clearly $\gamma(G) \leq \gamma_{pr}(G)$ and $\gamma(G) \leq \gamma_{dd}(G)$ for any graph G without isolated vertices for more comprehensive treatment of domination and for terminology not defined, here [3].

In this paper we use this idea to develop another new concept called paired-double dominating set and Paired-double domination number of a graph. A set S is called a paired-double dominating set if it is a double dominating set and the induced subgraph $\langle S \rangle$ contains at least one perfect matching. The minimum cardinality taken over all paired-double dominating sets is called the paired-double domination number and is denoted by γ_{prdd}. Any paired-double dominating set with γ_{prdd} vertices is called a γ_{prdd} set of G.

Theorem 1.1. *[5] For any non trivial tree* $\gamma_{pr}(T) \leq \gamma_{dd}(T)$.

Observation 1.1. *[5] For any graph* G*,*

1) *a support vertex is in every* $\gamma_{pr}(G)$ *set and in every* $\gamma_{dd}(G)$ *set.*
2) *a leaf is in every* $\gamma_{dd}(G)$ *set.*

Lemma 1.1. *[5] If* $\gamma_{pr}(T) = \gamma_{dd}(T)$ *for a tree* T*, then each support vertex of* T *is adjacent to exactly one leaf.*

2. Main Results

Theorem 2.1. *For any path* P_n*,* $\gamma_{prdd}(P_n) = \begin{cases} 2 & \text{if } n = 2 \\ \text{does not exist} & \text{if } n = 3 \\ 2\left\lfloor \frac{n}{3} \right\rfloor + 2 & \text{otherwise} \end{cases}$

Proof. The graph P_n contains n vertices and $n-1$ edges. Let $\{v_1, v_2, \ldots, v_{n-1}, v_n\}$ be the vertex set of P_n. Let S be the γ_{dd} set of P_n.

Case (i) $n = 2$.

The graph P_2 contains 2 vertices and 1 edge. Then $\gamma_{dd}(P_2) = 2$. Thus $\langle S \rangle$ contains one P_2 graph and has a perfect matching. Hence $\gamma_{prdd}(P_2) = 2$.

Case (ii) $n = 3$.

The P_3 graph contains 3 vertices and 2 edges. Then $S = \{v_1, v_2, v_3\}$ is a γ_{dd} set of P_3. Further more $\langle S \rangle$ has one independent edge. Which does not cover the set S. Thus $\langle S \rangle$ has no perfect matching. Hence $\gamma_{prdd}(P_3)$ does not exists.

Case (iii) if $n \geq 4$.

Then n is of the form $3k$ or $3k+1$ or $3k+2$. The proof is mathematical induction on the order k where $k \in \mathbb{N}$.

Sub case(i) $n = 3k$.

As the result fails when $k = 1$, we prove the result from $k = 2$. When $n = 6,\ k = 2$. To prove that the result is true for $n = 6$. If $k = 2$, then $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is the γ_{dd} set of P_6. It follows that $\langle S \rangle$ contains one P_6 graph. Thus $\langle S \rangle$ has a perfect matching. Hence $\gamma_{prdd}(P_6) = 6$. Therefore the result is true for $k = 2$. Now assume that the result is true for $k = l-1$. We have $\langle S \rangle$ of $P_{3(l-1)}$ contains $(l-3)P_2$ graph and one P_6 graph. That is $|V\langle S \rangle| = 2(l-3)+6$. To prove that the result is true for $k = l$. Let $n = 3k,\ k = l$, Then $S = \{v_1, v_2, v_3, v_4, v_5, v_6, v_8, v_9, \ldots, v_{3l-1}, v_{3l}\}$. It follows that $\langle S \rangle$ contains one P_6 graph and $(l-2)P_2$ graph. Hence $\gamma_{prdd}(P_{3l}) = \gamma_{prdd}(P_{3(l-1)}) + 2 = 2\left(\frac{3l-9}{3}\right) + 6 + 2 = 2l + 2 = 2k + 2$ (since $k = l$). $\gamma_{prdd}(P_n) = 2\left(\frac{3k}{3}\right) + 2 = 2\left\lfloor \frac{n}{3} \right\rfloor + 2$.

Sub case(ii) $n = 3k+1$ where $k \in \mathbb{N}$.

When $n = 4,\ k = 1$. To prove that the result is true for $k = 1$. If $n = 4,\ k = 1$, then $S = \{v_1, v_2, v_3, v_4\}$ is the γ_{dd} set of P_4. Here $\langle S \rangle$ contains one P_4 graph. Thus $\langle S \rangle$ has a perfect matching. Hence $\gamma_{prdd}(P_4) = 4$. Therefore the result is true for $k = 1$. Now assume that the result is true for $k = l-1$. We have $\langle S \rangle$ of $P_{3(l-1)+1}$ graph contains one P_4 graph and $(l-2)P_2$ graph. That is $|V\langle S \rangle| = 2(l-2)+4$. To prove that the result is true for $k = l$. Let $k = l,\ n = 3k+1$ where $k \in \mathbb{N}$. Then $S = \{v_1, v_2, v_3, v_4, v_6, v_7, \ldots, v_{3l}, v_{3l+1}\}$ is the γ_{dd} set of P_{3l+1}. It follows that $\langle S \rangle$ contains one P_4 graph and $(l-1)P_2$ graph. Hence $\gamma_{prdd}(P_{3l+1}) = \gamma_{prdd}(P_{3(l-1)+1}) + 2 = 2(l-2) + 4 + 2 = 2l - 4 + 4 + 2 = 2k + 2$(since $k = l$) $= 2\left(\frac{3k}{3}\right) + 2 = 2\left\lfloor \frac{3k+1}{3} \right\rfloor + 2 = 2\left\lfloor \frac{n}{3} \right\rfloor + 2$.

Sub case(iii) $n = 3k+2$ where $k \in \mathbb{N}$.

When $k = 1,\ n = 5$. To prove that the result is true for $k = 1$. If $n = 5,\ k = 1$, then $S = \{v_1, v_2, v_4, v_5\}$ is the γ_{dd} set of P_5. Here $\langle S\rangle$ contains one $2P_2$ graph. Thus $\langle S\rangle$ has a perfect matching. Hence $\gamma_{prdd}(P_5) = 4$. Therefore the result is true for $k = 1$. Now assume that the result is true for $k = l - 1$. We have $\langle S\rangle$ of $P_{3(l-1)+2}$ graph contains lP_2 graph. That is $|V\langle S\rangle| = 2(l)$. To prove that the result is true for $k = l$. Let $k = l,\ n = 3k + 2$ where $l \in \mathbb{N}$. Then $S = \{v_1, v_2, v_4, v_5, v_7, v_8, \ldots, v_{3l+1}, v_{3l+2}\}$ is the γ_{dd} set of P_{3l+2}. It follows that $\langle S\rangle$ contains $(l + 1)P_2$ graph. Hence $\gamma_{prdd}(P_{3l+2}) = \gamma_{prdd}(P_{3(l-1)+2}) + 2 = 2l + 2 = 2k + 2 = 2\left(\frac{3k}{3}\right) + 2 = 2\left\lfloor\frac{3k+2}{3}\right\rfloor + 2 = 2\left\lfloor\frac{n}{3}\right\rfloor + 2$. Thus from all the three cases we get $\gamma_{prdd}(P_n) = 2\left\lfloor\frac{n}{3}\right\rfloor$ when $n \geq 4$. □

Result 2.1. *Every support vertex incident with exactly one leaf and every support and leaves vertices must in every $\gamma_{dd}(G)$ and $\gamma_{prdd}(G)$ set.*

Theorem 2.2. *For any path P_n, $n \neq 3$ $\gamma_{dd}(P_n) \leq \gamma_{prdd}(P_n)$.*

Proof. Consider P_n let $\{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$ be the vertex set of P_n. The proof has three cases:

Case (i) $n \equiv 0(\text{mod } 3)$.

Take $n = 3k$ where $k \geq 2$. Let $S = \{v_1, v_2, v_4, v_5, v_7, v_8, \ldots, v_{3k-4}, v_{3k-2}, v_{3k-1}, v_{3k}\}$ be the γ_{dd} set of P_{3k} and M be the maximum matching of $\langle S\rangle$. Then $\gamma_{dd}(P_{3k}) = |S| = 2(k - 1) + 3$. Let B be the set of vertices incident to the edge set M. Let A be the set of vertices of S that are not saturated by M. Then $A = S - B$. Hence A has only one vertex. That is A has either v_{3k-2} or v_{3k}. Let $A = \{v_{3k}\}$. We know that v_{3k} is a leaf. By Result 2.2, every support and leaf vertices must in every $\gamma_{dd}(G)$ set and $\gamma_{prdd}(G)$ set. Therefore vertex v_{3k} must in M. We conclude that, A contains only one vertex say v_{3k-2}. Since S is a double dominating set, v_{3k-2} has one neighbor in B and another neighbor in $V - S$. Here $B \neq S$ and B does not double dominate V and hence we add a vertex say v_{3k-3} in S. We get, $\langle S\rangle$ contains one P_6 graph and $(k - 2)P_2$ graph. It follows that $\langle S\rangle$ has a perfect matching. Here $\gamma_{dd}(P_{3k}) = |S| < |S| + 1 = \gamma_{prdd}(P_{3k})$. Therefore $\gamma_{dd}(P_{3k}) < \gamma_{prdd}(P_{3k})$. Hence $\gamma_{dd}(P_n) < \gamma_{prdd}(P_n)$.

Case (ii) $n \equiv 1(\text{mod } 3)$.

Take $n = 3k+1$ where $k \geq 1$. Let $S = \{v_1, v_2, v_4, v_5, v_7, v_8, \ldots, v_{3k-4}, v_{3k-2}, v_{3k-1}, v_{3k}, v_{3k+1}\}$ be the γ_{dd} set of P_{3k+1} and M be the maximum matching of $\langle S\rangle$. Let B be the set of vertices incident to the edge set M. Now M is a perfect matching. Thus $B = S$ and B is a paired-double dominating set of P_{3k+1}. Hence

$\gamma_{dd}(P_{3k+1}) = |S| = \gamma_{prdd}(P_{3k+1})$. That is $\gamma_{dd}(P_n) = \gamma_{prdd}(P_n)$.
Case (iii) $n \equiv 2$(mod 3).

Take $n = 3k+2$ where $k \geq 1$. Let $S = \{v_1, v_2, v_4, v_5, v_7, v_8, \ldots, v_{3k-2}, v_{3k-1}, v_{3k+1}, v_{3k+2}\}$ be the γ_{dd} set of P_{3k+1} and M be the maximum matching of $\langle S \rangle$. Let B be the set of vertices incident to the edge set M. Now M is a perfect matching. Thus $B = S$ and B is a paired-double dominating set of P_{3k+2}. Hence $\gamma_{dd}(P_{3k+2}) = |S| = \gamma_{prdd}(P_{3k+2})$. That is $\gamma_{dd}(P_n) = \gamma_{prdd}(P_n)$. Thus form all the three cases, we get $\gamma_{dd}(P_n) \leq \gamma_{prdd}(P_n)$. □

Notation 2.1. *Let $\mathscr{L}(T)$ and $\mathscr{S}(T)$ denote the set of leaves and support vertices respectively of T.*

Theorem 2.3. *For any helm graph H_m, $\gamma_{prdd}(H_m) = 2m = \gamma_{dd}(H_m)$.*

Proof. The H_m graph contains $2m+1$ vertices and $3m$ edges. Let S be the γ_{dd} set of H_m and $\{v_1, v_2, v_3, \ldots, v_{2m+1}\}$ be the vertex set of H_m. Every $\gamma_{dd}(H_m)$ set S contains all the leaves and support vertices of H_m. Thus $\langle S \rangle$ has a perfect matching. Hence $\gamma_{dd}(H_m) = |\mathscr{L}(H_m)| + |\mathscr{S}(H_m)| = m + m = 2m = \gamma_{prdd}(H_m)$. □

Let H_m be a helm graph having centre w. Let P_n be any path with first vertex v_1. Consider a graph $H_m \cup P_n$. Let D be a graph obtained from $H_m \cup P_n$ by adding the edge wv_1.

Theorem 2.4. *Let D be the graph given in Figure 1. For any m if $n \equiv 1$(mod 3), $n \equiv 2$ (mod 3) and $\gamma_{prdd}(D) = \gamma_{dd}(D)$, then $\gamma_{prdd}(P_n) = \gamma_{dd}(P_n)$ and $\gamma_{dd}(D) = \gamma_{prdd}(D) = \gamma_{dd}(H_m) + \gamma_{dd}(P_n) = \gamma_{prdd}(H_m) + \gamma_{prdd}(P_n)$.*

Proof. Given $n \equiv 1$(mod 3), $n \equiv 2$(mod 3) and $\gamma_{prdd}(D) = \gamma_{dd}(D)$. Then from Theorem 2.3, $\gamma_{prdd}(P_n) = \gamma_{dd}(P_n)$. It is straight forwarded to see that $\gamma_{prdd}(H_m) = \gamma_{dd}(H_m) = |\mathscr{S}(H_m)| + |\mathscr{L}(H_m)| = 2|\mathscr{S}(H_m)|$.
Let S be the $\gamma_{dd}(D)$ set.
Case (i) If $w \notin S$, then $\gamma_{dd}(H_m) + \gamma_{dd}(P_n) = \gamma_{prdd}(H_m) + \gamma_{prdd}(P_n) = \gamma_{dd}(D) = \gamma_{prdd}(D)$. Therefore the result is true.
Case (ii) If $w \in S$, then $\gamma_{prdd}(H_m)$ does not exists. Let $S' = \{S - w\} \cup \{v_2\}$ where $v_2 \in N[v_1] - S$. Then $w \notin S'$ where S' is the γ_{dd} set of D. Now the proof is similar to Case (i). Hence the theorem. □

Theorem 2.5. *For any cycle C_n, $\gamma_{prdd}(C_n) = 2\left\lceil \frac{n}{3} \right\rceil$.*

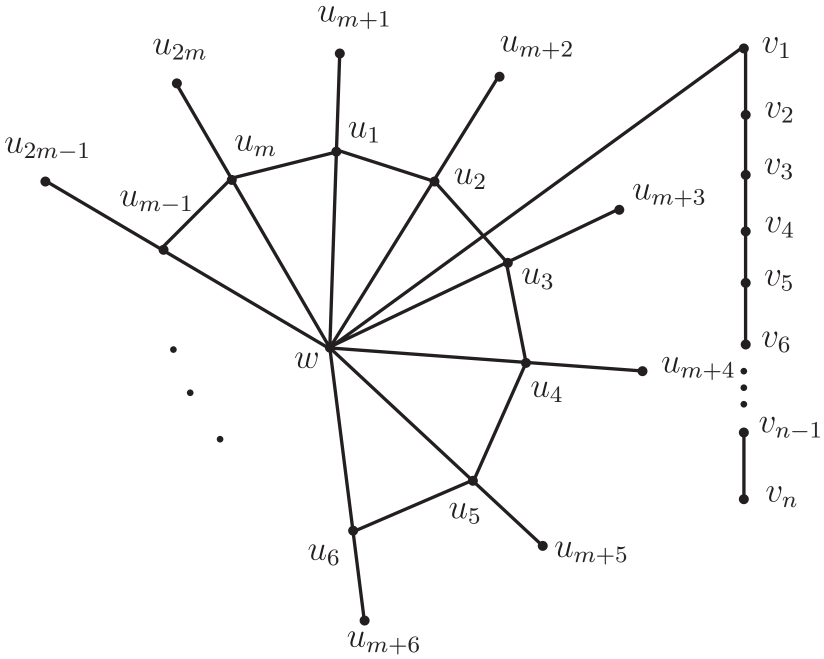

FIGURE 1

Proof. The graph C_n contains n vertices and n edges. Let $\{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$ be the vertex set of C_n and S be the γ_{dd} set of C_n.

Case (i) $n = 4$.

The graph C_4 contains 4 vertices and 4 edges. Then $S = \{v_1, v_2, v_3, v_4\}$ is the γ_{dd} set of C_4. Thus $\langle S \rangle$ is same as the graph C_4 and it has a perfect matching. Hence $\gamma_{prdd}(C_4) = 4$.

Case (ii) Let $n \geq 3,\ n \neq 4$.

Then n is of the form $3k$ or $3k+1$ or $3k+2$. The proof is mathematical induction on the order k where $k \in \mathbb{N}$.

Sub case(i) $n = 3k$ where $k \geq 1$. When $k = 1,\ n = 3$. To prove that the result is true for $k = 1$. If $k = 1$, then $S = \{v_1, v_2\}$ is the γ_{dd} set of C_3. It follows that $\langle S \rangle$ contains one P_2 graph. Thus $\langle S \rangle$ has a perfect matching. Hence $\gamma_{prdd}(C_3) = 2$. Therefore the result is true for $k = 1$. Now assume that the result is true for $k = l - 1$. We have $\langle S \rangle$ of $C_{3(l-1)}$ contains $(l-1)P_2$ graph that is $|V\langle S \rangle| = 2(l-1)$. To prove that the result is true for $k = l$. Let $n = 3k,\ k = l$. Then $S = \{v_1, v_2, v_4, v_5, v_7, v_8, \ldots, v_{3l-5}, v_{3l-4}, v_{3l-2}, v_{3l-1}\}$ is the γ_{dd} set of C_{3k}. It follows that $\langle S \rangle$ contains lP_2 graph. Thus $\langle S \rangle$ has a perfect matching. Hence

$\gamma_{prdd}(C_{3l}) = \gamma_{prdd}(C_{3(l-1)}) + 2 = 2\left(\frac{3l-3}{3}\right) + 2 = 2\left(\frac{3k}{3}\right)$ (since $k = l$) $= 2\left\lceil\frac{n}{3}\right\rceil$.

Sub case(ii) $n = 3k + 1$ where $k \in \mathbb{N},\ n \neq 4$.

When $k = 2,\ n = 7$. To prove that the result is true for $k = 2$. If $n = 7,\ k = 2$, then $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is the γ_{dd} set of C_7. It follows that $\langle S\rangle$ contains one P_6 graph. Thus $\langle S\rangle$ has a perfect matching. Hence $\gamma_{prdd}(C_7) = 6$. Therefore the result is true for $k = 2$. Now, assume that the result is true for $k = l - 1$. We have $\langle S\rangle$ of $C_{3(l-1)+1}$ contains one P_6 graph and $(l-3)P_2$ graph. That is $|V\langle S\rangle| = 6 + 2(l-3)$. To prove that the result is true for $k = l$. Let $k = l,\ n = 3k + 1$. Then $S = \{v_1, v_2, v_3, v_4, v_5, v_6, \ldots, v_{3l-4}, v_{3l-3}, v_{3l-1}, v_{3l}\}$ is the γ_{dd} set of C_{3k+1}. It follows that $\langle S\rangle$ contains one P_6 graph and $(l-2)P_2$ graph. Hence $\gamma_{prdd}(C_{3l+1}) = \gamma_{prdd}(C_{3(l-1)+1}) + 2 = 2\left(\frac{3l-9}{3}\right) + 6 + 2 = 2\left(\frac{3l}{3}\right) + 2 = 2\left(\frac{3k}{3}\right)$ (since $k = l$) $= 2\left\lceil\frac{3k+1}{3}\right\rceil = 2\left\lceil\frac{n}{3}\right\rceil$.

Sub case(iii) $n = 3k + 2$ where $k \in \mathbb{N}$.

When $k = 1,\ n = 5$. To prove that the result is true for $k = 1$. If $n = 5,\ k = 1$, then $S = \{v_1, v_2, v_3, v_4\}$ is the γ_{dd} set of C_5. Here $\langle S\rangle$ contains one P_4 graph. Thus $\langle S\rangle$ has a perfect matching. Hence $\gamma_{prdd}(C_5) = 4$. Therefore the result is true for $k = 1$. Now assume that the result is true for $k = l - 1$. We have $\langle S\rangle$ of $C_{3(l-1)+2}$ graph contains one P_4 graph and $(l-2)P_2$ graph. To prove that the result is true $k = l$. Let $k = l,\ n = 3k + 2$ where $k \in \mathbb{N}$. Then $S = \{v_1, v_2, v_3, v_4, \ldots, v_{3l}, v_{3l+1}\}$. It follows that $\langle S\rangle$ contains P_4 graph and $(l-1)P_2$ graph. Hence $\gamma_{prdd}(C_{3l+2}) = \gamma_{prdd}(C_{3(l-1)+2}) + 2 = 2\left(\frac{3k-6}{3}\right) + 4 + 2 = 2(l-2) + 4 + 2 = 2l + 2 = 2(l+1) = 2(k+1)$ (since $k = l$) $= 2\left\lceil\frac{3k+2}{3}\right\rceil = 2\left\lceil\frac{n}{3}\right\rceil$. Hence from all the three cases we get $\gamma_{prdd}(G) = 2\left\lceil\frac{n}{3}\right\rceil$. □

Theorem 2.6. *If $n = 3k + 2$ where $k \in \mathbb{N}$, then $\gamma_{prdd}(P_n) = \gamma_{prdd}(C_n)$.*

Proof. If $n = 3k+2$, then $\gamma_{prdd}(P_n) = 2\left\lfloor\frac{n}{3}\right\rfloor + 2 = \left\lfloor\frac{3k+2}{3}\right\rfloor + 2 = 2k + 2 = 2(k+1) = 2\left\lceil\frac{3k+2}{3}\right\rceil = 2\left\lceil\frac{n}{3}\right\rceil = \gamma_{prdd}(C_n)$. □

Theorem 2.7. *For any cycle C_n, $\gamma_{dd}(C_n) \leq \gamma_{prdd}(C_n)$.*

Proof. Consider C_n. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of C_n. The proof has three cases:

Case (i) $n \equiv 0 (\text{mod } 3)$.

Take $n = 3k$ where $k \geq 1$. Let $S = \{v_1, v_2, v_4, v_5, v_7, v_8 \ldots, v_{3k-2}, v_{3k-1}\}$ be the γ_{dd} set of C_{3k} and M be the maximum matching of $\langle S\rangle$. Let B be the set of vertices incident to the edge set M. Now M is a perfect matching. Thus

$B = S$ and B is a paired-double dominating set of C_{3k}. Hence $\gamma_{dd}(C_{3k}) = |S| = \gamma_{prdd}(C_{3k})$. That is $\gamma_{dd}(C_n) = \gamma_{prdd}(C_n)$.

Case (ii) $n \equiv 1$(mod 3).

Take $n = 3k + 1$ where $k \geq 1$. Let $S = \{v_1, v_2, v_4, v_5, v_7, v_8 \ldots, v_{3k-2}, v_{3k-1}, v_{3k+1}\}$ be the γ_{dd} set of C_{3k+1} and M be the maximum matching of $\langle S \rangle$. Then $\gamma_{dd}(C_{3k+1}) = |S|$. Let B be the set of vertices incident to the edge set M. Let A be the set of all vertices of S that are not saturated by M. Then $A = S - B$. Hence A has only one vertex. That is A has either v_{3k+1} or v_2.

Sub case(i) Let $A = \{v_{3k+1}\}$. Then $\langle S \rangle$ has no perfect matching and $B \neq S$.Thus we add either a vertex say v_{3k} or v_3 in S. We get $\langle S \rangle$ contains one P_6 graph and $(k-2)P_2$ graph. It follows that $\langle S \rangle$ has a perfect matching. Here $\gamma_{dd}(C_{3k+1}) = |S| < |S| + 1 = \gamma_{prdd}(C_{3k+1})$. Therefore $\gamma_{dd}(C_{3k+1}) < \gamma_{prdd}(C_{3k+1})$. Hence $\gamma_{dd}(C_n) < \gamma_{prdd}(C_n)$.

Sub case(ii) Let $A = \{v_2\}$. Then the proof is similar to case(i).

Case (iii) $n \equiv 2$(mod 3).

Take $n = 3k + 2$ where $k \geq 1$. Let $S = \{v_1, v_2, v_4, v_5, v_7, v_8 \ldots, v_{3k-2}, v_{3k-1}, v_{3k+1}, v_{3k+2}\}$ be the γ_{dd} set of C_{3k+2} and M be the maximum matching of $\langle S \rangle$. Let B be the set of vertices incident to the edge set M. Now M is a perfect matching. Thus $B = S$ and B is a paired-double dominating set of C_{3k+2}. Hence $\gamma_{dd}(C_{3k+2}) = |S| = \gamma_{prdd}(C_{3k+2})$. That is $\gamma_{dd}(C_n) = \gamma_{prdd}(C_n)$. Hence from all the three cases we get $\gamma_{dd}(C_n) \leq \gamma_{prdd}(C_n)$. □

Paired-double domination number for some standard graphs are given below

1) For the complete bipartite graph of order $p \geq 4$, $\gamma_{prdd}(K_{m,n}) = 4$ (where $m, n \geq 2$ and $m + n = p$).
2) For any complete graph of order $p \geq 4$, $\gamma_{prdd}(K_p) = 2$.
3) For any wheel graph of order $p \geq 4$, $\gamma_{prdd}(W_p) = 2\left\lceil \frac{p}{4} \right\rceil$.

REFERENCES

[1] J. A. BONDY, U. S. R. MURTY: *Graph theory*, Springer, 2008.

[2] F. HARARY, T. W. HAYNES: *Double domination in graphs*, Ars Combin., **55** (2000), 201–213.

[3] T. W. HAYNES, S. T. HEDETNIEMI, P. J. SLATER: *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[4] T. W. HAYNES, M. CHELLALI, M. BLIDIA: *Characterisation of trees with equal paired and double domination numbers*, 2013.

[5] T. W. HAYNES, P. J. SLATER: *Paired-domination in graphs*, Networks, **32** (1998), 199–206.

DEPARTMENT OF MATHEMATICS
S. T. HINDU COLLEGE
NAGERCOIL, TAMIL NADU
Email address: sreetharosh2014@gmail.com

DEPARTMENT OF MATHEMATICS
S. T. HINDU COLLEGE
NAGERCOIL, TAMIL NADU
Email address: anusuyameenu@yahoo.com