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RANK DECREASING GRAPH OF UPPER TRIANGULAR MATRIX RINGS OVER $\mathbb{Z}_P$

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ABSTRACT. Let R be a commutative ring with identity $1 \neq 0$. For a positive integer $n \geq 2$, let $T_n(R)$ denote the ring of $n \times n$ upper triangular matrices over the ring R and $T_n(R)^*$ denote the set of non-zero upper triangular matrices of $T_n(R)$. The Rank decreasing graph denoted by $\mathbb{T}(R)$, is a graph with $T_n(R)^*$ as the vertex set where two distinct vertices A and B are adjacent if and only if $rank(AB) < rank(B)$, where $A, B \in T_n(R)^*$. Here we study some graph theoretic properties of rank decreasing graph for $\mathbb{T}$ over $\mathbb{Z}_p$. Then we investigate the relation between the graph theoretic properties of $\mathbb{T}(T_n(\mathbb{Z}_p))$ and the ring theoretic properties of $T_n(\mathbb{Z}_p)$.

1. INTRODUCTION

Let R be a commutative ring with identity $1 \neq 0$. For a positive integer $n \geq 2$, let $T_n(R)$ denote the ring of $n \times n$ upper triangular matrices over the ring R and $T_n(R)^*$ denote the set of non-zero upper triangular matrices of $T_n(R)$. S Redmond [2] introduced the concept of zero divisor graph of non commutative rings. The zero divisor graph $\Gamma(R)$ over a non commutative ring R is a graph with vertex set $Z(R)^*$ and there is a directed edge from a vertex x to a distinct vertex y if and only if $x.y = 0$. Akbari and Mohammadian [3] studied the problem of determining when the zero divisor graph of rings are isomorphic.

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Aihua Li and Ralph P Tucci [4] studied the zero divisor graph of upper triangular matrix rings.

Dengyin Wang, Xiaobin Ma and Fengli Tian [5] introduced the rank decreasing graph over the semigroup of upper triangular matrices . They denoted the rank decreasing graph as $\mathbb{T}$ and defined as follows: the vertex set of $\mathbb{T}$ is $T_n(F_p)^*$ and there is an directed edge from A to B (not distinct) if and only if $rank(AB) < rank(B)$, where $A, B \in T_n(F_p)^*$ and $rank(A)$ denote the rank of the matrix A. A loop exists at A if $rank(A^2) < rank(A)$.

2. Preliminaries and Basic Results

A matrix A over a ring R is usually denoted as $A = (a_{ij})$. E_{ij} [6] represents the matrix with 1 as the ij^{th} entry and 0 elsewhere. For a directed graph $D = (V, E)$, [1] $d^+(v)$ denotes the outdegree of a vertex $v \in V$ and $d^-(v)$ denotes the indegree of a vertex $v \in V$. Throughout this paper, we refer to $\mathcal{T}_n(\mathbb{Z}_p)$ as the zero divisor graph of $T_n(R)$. The definitions enlisted below are useful for subsequent reading.

A vertex v of a digraph D is said to be reachable [1] from a vertex u if there is a (u, v) dipath in D.

A digraph D is said to be weakly connected or simply connected [1] if its underlying graph is connected.

For a digraph D, if for any pair of vertices u and v, u is reachable from v and v is reachable from u, then D is strongly connected or disconnected [1].

The girth [1] of D is the length of a shortest cycle in D, and the digirth is the length of the shortest directed cycle in D.

A tournament [1] is an orientation of a complete graph.

A graph G is said to sub-Eulerian [6] if it is a spanning subgraph of some Eulerian graph and a non Eulerian graph G is said to be super-Eulerian [6] if it has a spanning Eulerian subgraph.

A point basis [1] of D is a minimal collection of points from which all points are reachable.

A 1-basis [1] is a minimal collection S of mutually nonadjacent points such that every point of D is either in S or adjacent from a point of S.

Definition 2.1. *[4] We denote $E(i, j)$ as the set of all directed edges from vertices in $\Gamma(T^i)$ to vertices in $\Gamma(T^j)$.*

Remark 2.1. *[4] $E(i,j) \subseteq T^i \times T^j$, where $T^i \times T^j$ is denoted as follows:*

$$T^i \times T^j = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a,b,c \in \mathbb{Z}_p \right\} \times \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} : x,y,z \in \mathbb{Z}_p \right\}.$$

The equality occurs if there is an edge from every vertex in T^i to every vertex in T^j.

Theorem 2.1. *[4] Let R be an integral domain and $\Gamma(T_2(R))$ be the zero divisor graph. Consider T^i and T^j are defined above. Then $T^i \times T^j \neq \phi$ if and only if $(i,j) \in P$, where*

$$P = \{(0,0), (0,1), (0,3), (1,2), (2,0), (2,1), (2,3), (3,4), (4,0), (4,1), (4,3)\}.$$

Theorem 2.2. *[4] Let R be an integral domain and $\Gamma(T_2(R))$ be the zero divisor graph.*

(1) *If $(i,j) \in P$ but $(i,j) \neq (3,4)$ then $E(i,j) = T^i \times T^j$.*
(2) $E(\mathbb{T}(T_2(\mathbb{Z}_p))) = E(3,4) \cup \sum_{(i,j) \in P/\{3,4\}} T^i \times T^j$.

3. Rank Decreasing Graph of the Ring of Upper Triangular Matrices over $\mathbb{Z}_p$

3.1. **Basic Structure of $\mathbb{T}(T_2(\mathbb{Z}_p))$.** In this section, we discuss basic structure of the rank decreasing graph of the upper triangular matrix rings over $\mathbb{Z}_p$. We analyse the degree of the vertices of this graph and its connectedness.

Observation 3.1. *The zero divisor graph $\mathcal{T}_2(\mathbb{Z}_p)$ is a subgraph of the rank decreasing graph of $T_2(\mathbb{Z}_p)$. i.e., $\mathcal{T}_2(\mathbb{Z}_p) \subset \mathbb{T}(T_2(\mathbb{Z}_p))$, since if $A \in V(\mathcal{T}_2(\mathbb{Z}_p))$ then there exits B in $T(\mathbb{Z}_p)$ such that $A.B = 0$. Therefore $rank(AB) < rank(B)$, thus AB is an edge in $\mathbb{T}(T_2(\mathbb{Z}_p))$.*

To characterize the rank decreasing graph $\mathbb{T}(T_2(\mathbb{Z}_p))$, we first classify the non zero elements of $T_2(\mathbb{Z}_p)$ into the following disjoint subsets:

$$T^0 = \left\{ \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} : u,v \in \mathbb{Z}_p^* \right\}, \qquad T^1 = \left\{ \begin{bmatrix} u & v \\ 0 & w \end{bmatrix} : u,v,w \in \mathbb{Z}_p^* \right\},$$

$$T^2 = \left\{ \begin{bmatrix} u & v \\ 0 & 0 \end{bmatrix} : u,v \in \mathbb{Z}_p^* \right\}, \qquad T^3 = \left\{ \begin{bmatrix} 0 & u \\ 0 & v \end{bmatrix} : u,v \in \mathbb{Z}_p^* \right\},$$

$$T^4 = \left\{ \begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} : u \in \mathbb{Z}_p^* \right\}, \qquad T^5 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & u \end{bmatrix} : u \in \mathbb{Z}_p^* \right\},$$

$$T^6 = \left\{ \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} : u \in \mathbb{Z}_p^* \right\}$$

The vertex set of $\mathbb{T}(T_2(\mathbb{Z}_p))$ is $T^0 \cup T^1 \cup T^2 \cup T^3 \cup T^4 \cup T^5 \cup T^6$, the disjoint union of the seven sets T^is for $i = 0, 1, \ldots, 6$. To discuss the structure of $\mathbb{T}(T_2(\mathbb{Z}_p))$, we first define the related edge sets.

Definition 3.1. *We denote $E(i, j)$ as the set of all directed edges from vertices in $\mathbb{T}(T^i)$ to vertices in $\mathbb{T}(T^j)$, where $i, j \in \{0, 1, 2, 3, 4, 5, 6\}$.*

Proposition 3.1.

(1) $E(2, j) = T^2 \times T^j = \{A \times B : A \in T^2, B \in T^j\}$*, where* $j \in \{0, 1\}$.
(2) $E(3, j) = T^3 \times T^j = \{A \times B : A \in T^3, B \in T^j\}$*, where* $j \in \{0, 1, 2, 4, 6\}$.
(3) $E(4, j) = T^4 \times T^j = \{A \times B : A \in T^4, B \in T^j\}$*, where* $j \in \{0, 1, 5\}$.
(4) $E(5, j) = T^5 \times T^j = \{A \times B : A \in T^2, B \in T^j\}$*, where* $j \in \{0, 1, 2, 4, 6\}$.
(5) $E(6, j) = T^6 \times T^j = \{A \times B : A \in T^2, B \in T^j\}$*, where* $j \in \{0, 1, 2, 4, 6\}$.

Proof.

(1) Let $A = \begin{bmatrix} u_1 & v_1 \\ 0 & 0 \end{bmatrix} \in T^2$ and $B = \begin{bmatrix} u_2 & v_2 \\ 0 & w_2 \end{bmatrix} \in T^2$. Then

$$AB = \begin{bmatrix} u_1.u_2 & u_1.v_2 + v_1.w_2 \\ 0 & 0 \end{bmatrix},$$

which implies that $rank(AB) < rank(B)$. Hence the result.

(2) Let $A = \begin{bmatrix} 0 & u_1 \\ 0 & v_1 \end{bmatrix} \in T^3$ and $B = \begin{bmatrix} u_2 & v_2 \\ 0 & 0 \end{bmatrix} \in T^2$. Then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which implies that $rank(AB) < rank(B)$. Hence the result.

Proof of the remaining parts follows from part 1 and part 2 $\square$

Lemma 3.1. *Let $\mathbb{T}(T_2(\mathbb{Z}_p))$ be the rank decreasing graph which is defined and T^i and T^j are mentioned in the above context. Then*

$$T^i \times T^j \neq \phi$$

if and only if $(i, j) \in P$*, where* $P = \{(6,6), (6,4), (6,0), (6,1), (6,2), (5,0), (5,1), (5,2), (5,6), (5,4), (4,5), (4,1), (4,0), (3,0), (3,1), (3,6), (3,4), (3,2), (2,1), (2,0), (2,3)\}$.

Observation 3.2. *The number of directed edges from T^2 to T^3 is given as follows: An edge in $E(2,3)$ has the form $A \to B$ where $A = \begin{bmatrix} u_1 & u_2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & v_1 \\ 0 & v_2 \end{bmatrix}$ with $u_1, u_2, v_1, v_2 \in \mathbb{Z}_p^*$ with $rank(AB) < rank(B)$. We know that if $rank(AB) = 0$ then $rank(AB) < rank(B)$. This implies that $AB = 0 \Rightarrow u_1.v_1 + u_2.v_2 = 0$. Solving this equation, we get $v_1 = -u_1^{-1}u_2v_2$. Therefore the number of directed edges from T^2 to T^3 is $(p-1)^3$.*

Observation 3.3. *The rank decreasing graph of $T_2(\mathbb{Z}_p)$ are digraphs. They are also multigraphs because for any two matrices $A, B \in T_2(\mathbb{Z}_p)$, $rank(AB) \neq rank(BA)$.*

Theorem 3.1. *Let $\mathbb{T}(T_2(\mathbb{Z}_p))$ be the rank decreasing graph which is defined above and $E(i.j)$ as the set of all directed edges A to B, where $A \in T^i, B \in T^j$.*

(1) *If $(i,j) \in P$ but $(i,j) \neq (2,3)$ then $E(i,j) = T^i \times T^j$.*
(2) $E(\mathbb{T}(T_2(\mathbb{Z}_p))) = E(2,3) \cup \sum_{(i,j)\in P/\{2,3\}} T^i \times T^j$.
(3) $|E(\mathbb{T}(T_2(\mathbb{Z}_p)))| = (p-1)^2(2p^3+2p+1)$.

Proof.

(1) By Lemma 3.1, there are twenty non empty $E(i,j)$ other than $E(2,3)$. This $E(i,j)$'s are of the form $E(i,j) = T^i \times T^j$.
(2) It follows from part 1 and observation 3.2.
(3) $E(\mathbb{T}(T_2(\mathbb{Z}_p))) = E(2,3) \cup \sum_{(i,j)\in P/\{2,3\}} T^i \times T^j$, where $P = \{(6,6),(6,4),(6,0),(6,1),(6,2),(5,0),(5,1),(5,2),(5,6),(5,4),(4,5),(4,1),(4,0),(3,0),(3,1),(3,6),(3,4),(3,2),(2,1)\,(2,0)\}$.

Thus $|E(\mathbb{T}(T_2(\mathbb{Z}_p)))| = |E(2,3)| + \sum_{(i,j)\in P/\{2,3\}} |T^i|.|T^j|$. It is easy to calculate the following:
$|E(6,4)| = |E(6,6)| = |E(5,6)| = |E(5,4)| = |E(4,5)| = (p-1)^2$.
$|E(6,1)| = |E(6,2)| = |E(5,1)| = |E(5,2)| = |E(4,1)| = |E(3,6)| = |E(3,4)| = (p-1)^3$.
$|E(6,0)| = |E(5,0)| = |E(4,0)| = |E(3,1)| = |E(3,2)| = |E(2,1)| = (p-1)^4$.
$|E(3,0)| = |E(2,0)| = (p-1)^5$.
Therefore,

$$|E(\mathbb{T}(T_2(\mathbb{Z}_p)))| = 5(p-1)^2 + 8(p-1)^3 + 6(p-1)^4 + 2(p-1)^5$$
$$= (p-1)^2(2p^3+2p+1).$$

□

Theorem 3.2. *$\mathbb{T}(T_2(\mathbb{Z}_p))$ is simply connected.*

Proof. We have to prove that $\mathbb{T}(T_2(\mathbb{Z}_p))$ is simply connected. Thus, by definition, it is enough to prove that the underlying graph is connected. For any $A, B \in \mathbb{T}(T_2(\mathbb{Z}_p))$, there exits a path in $\mathbb{T}(T_2(\mathbb{Z}_p))$. This implies that $rank\,(AB) = 0$ or 1 or 2. We examine each of the following cases:
Case 1: If $rank\,(AB) = 0$ then AB is an edge in $\mathbb{T}(T_2(\mathbb{Z}_p))$.
Case 2: If $rank\,(AB) = 1$, then we have two cases: $rank(B) = 1$ or 2. If $rank\,(B) = 2$, then AB is an edge in $\mathbb{T}(T_2(\mathbb{Z}_p))$. If $rank\,(B) = 1$, then there exists a $C \in \mathbb{T}(T_2(\mathbb{Z}_p))$ such that $BC = 0$ and $CA = 0$. Thus $B - C - A$ form a path of length 2.
Case 3: If $rank\,(AB) = 2$, then we have two cases : $rank(B) = 1$ or 2. If $rank\,(B) = 1$, then $rank\,(A) = 2$. Thus BA is an edge in $\mathbb{T}(T_2(\mathbb{Z}_p))$. If $rank\,(B) = 2$, then there exists a $C \in \mathbb{T}(T_2(\mathbb{Z}_p))$ such that $BC = 0$ and $CA = 0$. Thus $B - C - A$ form a path of length 2. □

Theorem 3.3. *For any vertex A of $\mathbb{T}(T_2(Z_p))$,*

$$d^+(A) = \begin{cases} 0 & if\ rank\,(A^2) = 2 \\ p^3 - p^2 + 2p - 2 & if\ rank\,(A^2) = 0 \end{cases},$$

$$d^-(A) = \begin{cases} 2p^2 - p - 1 & if\ rank\,(A^2) = 2 \\ p^2 + p - 2 & if\ rank\,(A^2) = 0 \end{cases}.$$

Proof.
Case 1 : If $rank\,(A^2) = 2$, then $rank\,(A) = 2$. Let $B \in T_2(Z_p)$. Here we consider two subcases : $rank(B) = 1$ or 2.
If $rank\,(B) = 2$, then $rank\,(AB) = 2$. Thus there are no arcs from A to B and B to A.
If $rank\,(B) = 1$, then $rank\,(BA) < rank\,(A)$ but $rank\,(B) < rank\,(AB)$. Therefore $d^+(A) = 0$.
The sets T^i, where $i = 2, 3, 4, 5, 6$ contains rank 1 matrices. Then the matrix $B \in T^i$, where $i \in \{2, 3, 4, 5, 6\}$ is adjacent to $A \in T^j$, where $j \in \{0, 1\}$. Thus there is an arc from B to A but no arc from A to B.

Therefore $|T^i| = p-1$, where $i \in \{4,5,6\}$ and $|T^i| = (p-1)^2$, where $i \in \{2,3\}$.

$$\begin{aligned} d^-(A) &= 3(p-1) + 2(p-1)^2 \\ &= 2p^2 - p - 1. \end{aligned}$$

Therefore $d^-(A) = 2p^2 - p - 1$, (A is adjacent to all other vertices in $\mathbb{T}$, since $rank\,(BA) < rank\,(A)$)

Case 2 : If $rank\,(A^2) = 0$, then $rank\,(A) = 1$. We know that A is a nilpotent element, then $A.B = 0$ OR $B.A = 0$. We examine each of the following cases.

Subcase 2.1: If $A.B = 0$, then $B \in T^i$, where $i \in \{0,1,2,4,6\}$. Therefore $|T^i| = (p-1)^3$, where $i \in \{0\}$, $|T^i| = (p-1)^2$, where $i \in \{1,2\}$ and $|T^i| = p-1$, where $i \in \{4,6\}$.

$$\begin{aligned} d^-(A) &= 3(p-1) + 2(p-1)^2 + (p-1)^3 \\ &= p^3 - p^2 + 2p - 2. \end{aligned}$$

Subcase 2.2: If $B.A = 0$,then $B \in T^i$, where $i \in \{3,5,6\}$. $|T^i| = (p-1)^2$, where $i \in \{3\}$ and $|T^i| = p-1$, where $i \in \{5,6\}$.

$$\begin{aligned} d^+(A) &= 3(p-1) + (p-1)^2 \\ &= p^2 + p - 2 \end{aligned}$$

$d^+(A) = p^3 - p^2 + 2p - 2$ and $d^-(A) = p^2 + p - 2$. □

Corollary 3.1. *For any vertex A of $\mathbb{T}(T_2(Z_p))$ with $rank\,(A^2) = 1$*

$$d^+(A) = \begin{cases} 2p^3 - 5p^2 + 4p - 1 & if\;\; A \in T^2 \\ p^3 - p^2 + p - 1 & if\;\; A \in T^i, where\; i = 3\; or\; 5 \\ p^3 - 2p^2 - 2p - 1 & if\;\; A \in T^4 \end{cases}$$

and

$$d^-(A) = \begin{cases} p - 1 & if\;\; A \in T^5 \\ p^2 - 1 & if\;\; A \in T^i, where\; i = 2\; or\; i = 4. \\ (p-1)^3 & if\;\; A \in T^3 \end{cases}$$

Theorem 3.4. *Girth of $\mathbb{T}(T_2(\mathbb{Z}_p))$ is 3, and the digirth is also 3.*

Proof. The vertices in $\mathbb{T}(T_2(\mathbb{Z}_p))$ $\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$ - $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ - $\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}$ - $\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$ form a directed cycle in $\mathbb{T}(T_2(\mathbb{Z}_p))$. Therefore, the digirth of $\mathbb{T}(T_2(\mathbb{Z}_p))$ is 3. □

3.2. **Eulerian Properties of Rank Decreasing Graph.** *In this section, we discuss the Eulerian properties of the rank decreasing graph. Clearly, the graph is not Eulerian, however, there are some interesting Eulerian properties of the graph such as sub-Eulerian, Super-Eulerian etc.*

Theorem 3.5. $\mathbb{T}(T_2(\mathbb{Z}_p))$ *is not Eulerian.*

Proof. In order to show that $\mathbb{T}(T_2(\mathbb{Z}_p))$ is not Eulerian, we have to find a vertex $A \in \mathbb{T}(T_2(\mathbb{Z}_p))$ in which $d^+(A) = d^-(A)$. By theorem 2.2, no vertices in $\mathbb{T}(T_2(\mathbb{Z}_p))$ have same in degree and out degree. Therefore, $\mathbb{T}(T_2(\mathbb{Z}_p))$ is not Eulerian. ☐

Theorem 3.6. $\mathbb{T}(T_2(\mathbb{Z}_p))$ *is super Eulerian.*

Proof. We have to prove that $\mathbb{T}(T_2(\mathbb{Z}_p))$ is super Eulerian. It is enough to show that $\mathbb{T}(T_2(\mathbb{Z}_p))$ contains a spanning Eulerian subgraph. The vertices $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ and $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ form a Eulerian subdigraph in $\mathbb{T}(T_2(\mathbb{Z}_p))$. Thus $\mathbb{T}(T_2(\mathbb{Z}_p))$ is super Eulerian. ☐

3.3. **Advanced Properties of Rank Decreasing Graph.** *In this section, we discuss the number of tournaments of the rank decreasing graph. Then we discuss the digraph properties which include point basis and 1- basis of digraph.*

Theorem 3.7. *The order of tournament in* $\mathbb{T}(T_2(\mathbb{Z}_p))$ *is* $p-1$*, where* $p > 2$.

Proof. The vertices of the form $T^6 = \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}$, where u is the unit in $\mathbb{Z}_p$ form a complete induced subgraph of $\mathbb{T}(T_2(\mathbb{Z}_p))$. Then the cardinality of T^6 is $p-1$. Therefore the order of tournaments in $\mathbb{T}(T_2(\mathbb{Z}_p))$ is $p-1$. ☐

Theorem 3.8. *Let* $H = \{A : A \in tournament\ of\ order\ p-1\ in\ \mathbb{T}(T_2(\mathbb{Z}_p))\}$.*Then* $H \cup \{[0]_{2\times 2}\}$ *is an ideal in* $T_2(\mathbb{Z}_p)$.

Proof. Assume that the vertices form a tournament in $\mathbb{T}(T_2(\mathbb{Z}_p))$. Then the vertices of $\mathbb{T}(T_2(\mathbb{Z}_p))$ are contained in $T^6 = \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}$. We have to show that $T^6 \cup \{[0]_{2\times 2}\}$ is a non trivial ideal of the ring $T_2(\mathbb{Z}_p)$. Let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in T_2(\mathbb{Z}_p)$.

Then $AB = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \cdot \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & au \\ 0 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & cu \\ 0 & 0 \end{bmatrix}$, where $B = \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} \in T^6$. Thus $T^6 \cup \{[0]_{2\times 2}\}$ is an ideal in in $T_2(\mathbb{Z}_p)$. □

Theorem 3.9. *Let* $\mathcal{B} = \{A : rank\,(A) = 1,\ A\ is\ not\ adjacent\ to\ B, where\ A, B \in V(\mathbb{T}(T_2(\mathbb{Z}_p)))\}$ *then* $\mathcal{B}$ *is a 1-basis in* $\mathbb{T}(T_2(\mathbb{Z}_p))$.

Proof. Let V be the collection of all vertices in $\mathbb{T}(T_2(\mathbb{Z}_p))$ and S be the collection of all non adjacent vertices in $\mathbb{T}(T_2(\mathbb{Z}_p))$. Then $S = \{T^3, T^5\}$. We have to show that these elements form a 1-basis of the digraph. We know that

$$T^3 = \left\{ \begin{bmatrix} 0 & u \\ 0 & v \end{bmatrix} : u, v \in \mathbb{Z}_p^* \right\} \quad \text{and} \quad T^5 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & u \end{bmatrix} : u \in \mathbb{Z}_p^* \right\}.$$

Let $A \in T^3$, then $rank\,(AB) = rank\,(B)$ and $rank\,(BA) = rank\,(A)$, for any $B \in T^5$. Thus T^3 and T^5 are non adjacent.

$V/S = T^i$, where $i \in \{0, 1, 3, 5, 6\}$. Let $A \in T^5$. $rank\,(AB) < Rank\,(B)$, where $B \in T^i, i = \{0, 1, 2, 6\}$. Thus there is a directed arc from A to B.

Let $A \in T^3$. $rank\,(AB) < Rank\,(B)$, where $B \in T^i, i = \{2, 4, 6\}$. Thus there is a directed arc from A to B. Thus $S = \{T^3, T^5\}$ form an 1-basis for $\mathbb{T}(T_2(\mathbb{Z}_p))$. □

Observation 3.4. *Every digraph has a point basis, but every digraph does not have a 1-basis.* $\mathbb{T}(T_2(\mathbb{Z}_p))$ *has 1-point basis.*

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