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# THE STUDY OF BERWALD CONNECTION OF A FINSLER SPACE WITH A SPECIAL $(\alpha, \beta)$-METRIC

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ABSTRACT. We deal with one of the special $(\alpha, \beta)$-metrics and investigate the necessary and sufficient conditions for the Finsler space to be Berwald space, where $\alpha$ is a Riemannian metric and $\beta$, a differential 1-form and also we obtain the vector field in the Berwald connection of a Finsler space with special $(\alpha, \beta)$-metric. Furthermore, we provide an example to illustrate the result.

## 1. INTRODUCTION

L. Berwald introduced a connection and two curvature tensors in 1926. T. Okada [7] proved that the Berwald connection of a Finsler space is the h-connection, which is uniquely determined from the fundamental function by four axioms by taking a hint from J. Grifone [1]. Jingwei Han, Yao-yong Yu, Jing Yu proved the rigidity theorem [4] using Berwald connection on Finsler manifold $(M, F)$ of dim $n$ and Berwald manifold $(\tilde{M}, \tilde{F})$ of dim $m$ and considering a map with zero tension field. H. S. Park, H. Y. Park and B. D. Kim [3] obtained Berwald connection and concrete form of the Berwald connection in a Finsler space with a special $(\alpha, \beta)$-metric.

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## 2. Preliminaries

The concept of $(\alpha,\beta)$-metric $L(\alpha,\beta)$ was introduced by M. Matsumoto in 1972 and studied by many authors like [2], [6], [3]. Throughout the present paper, the terminology and notation are referred to Matsumoto's monograph [5]. Let $F^n = (M, L(\alpha,\beta))$ be an $n$-dimensional Finsler space with an $(\alpha,\beta)$-metric

$$L(\alpha,\beta) = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}, c_1 \neq 0, \tag{2.1}$$

where $\alpha$ is a Riemannian metric and $\beta$ is a differential 1-form. The Riemannian space $R^n = (M,\alpha)$ is called the associated Riemannian space with $F^n$ and the Christoffel symbols of $R^n = (M,\alpha)$ are indicated by $\gamma_j{}^i{}_k$. Then the Riemannian connection $\left(\gamma_j{}^i{}_k\right)$ gives rise to the linear Finsler connection $F\Gamma = (\gamma_j{}^i{}_k, \gamma_0{}^i{}_j, 0)$, where the subscript $0$ denotes a contraction by $y^i$.

The Berwald connection $B\Gamma = (G_j{}^i{}_k, G_0{}^i{}_j, 0)$ is a Finsler connection which is uniquely determined from the fundamental function $L(x,y)$ by the following Okada's axiomatic system( [7]):

(i) $L$-metrical: $L_{|i} = 0$,
(ii) $(h)h$-torsion tensor $T_j{}^i{}_k = G_j{}^i{}_k - G_k{}^i{}_j = 0$,
(iii) deflection tensor $D^i{}_j = y^k G_k{}^i{}_j - G^i{}_j = 0$,
(iv) $(v)hv$-torsion tensor $P_j{}^i{}_k = \dot{\partial}_k G^i{}_j - G_k{}^i{}_j = 0$,
(v) $(h)hv$-torsion tensor $C_j{}^i{}_k = 0$,

where the symbol $(|)$ in (i) denotes the $h$-covariant differentiation with respect to the Finsler connection.

Now, we shall find the Berwald connection $B\Gamma$ in $F^n$. Putting

$$2G^i = \gamma_0{}^i{}_0 + 2B^i, \tag{2.2}$$

we have from (ii),(iii) and (iv)

$$G^i{}_j = \dot{\partial}_j G^i = \gamma_0{}^i{}_j + B^i{}_j,$$
$$G_j{}^i{}_k = \dot{\partial}_j G^i{}_k = \gamma_j{}^i{}_k + B_j{}^i{}_k,$$

where we put $B^i{}_j = \dot{\partial}_j B^i$ and $B_j{}^i{}_k = \dot{\partial}_k B^i{}_j$.

The axiom (i): $L_{|i} = \dot{\partial}_i L - G^r{}_i \dot{\partial}_r L = 0$ is written as

$$L_\alpha B_j{}^k{}_i y^j y_k + \alpha L_\beta (B_j{}^r{}_i b_r - \nabla_i b_j) y^j = 0, \tag{2.3}$$

where $y_k = a_{ki} y^i$ and $\nabla_j$ is the differentiation with respect to $\gamma_j{}^i{}_k$.

## 3. THE STUDY OF BERWALD SPACE WITH SPECIAL $(\alpha, \beta)$-METRIC

A Finsler spae is called a Berwald space iff its Berwald connection is linear. In this section, we establish the following theorems:

**Theorem 3.1.** *Let $F^n$ be the Finsler space with a special $(\alpha,\beta)$-metric $L(\alpha,\beta) = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}$, $c_1 \neq 0$, with the Berwald connection $B\Gamma = (G_j{}^i{}_k, G_0{}^i{}_j, 0)$. Then we have the following:*

(i) *If $(c_2\beta^2 - \alpha^2) \neq 0$, then $F^n$ is a Berwald space if and only if $\nabla_i b_j = 0$ and the Berwald connection is $(\gamma_j{}^i{}_k, \gamma_0{}^i{}_j, 0)$.*

(ii) *If $(c_2\beta^2 - \alpha^2) = 0$, then $F^n$ is a Berwald space if and only if $B_j{}^k{}_i = 0$ and the Berwald connection is $(\gamma_j{}^i{}_k, \gamma_0{}^i{}_j, 0)$.*

*Proof.* We find the condition for $F^n$ to be a Berwald space by applying the Matsumoto's method of [6]. Since the metric function $L$ is given by (2.1), we get

$$L_\alpha = c_1 + \frac{2\alpha}{\beta},$$
$$L_\beta = c_2 - \frac{\alpha^2}{\beta^2}. \tag{3.1}$$

Substituting (3.1) in (2.3), we have

$$\begin{aligned} \alpha\{2\beta B_j{}^k{}_i y^j y_k &+ (c_2\beta^2 - \alpha^2)(B_j{}^r{}_i b_r - \nabla_i b_j) y^j\} \\ &+ c_1\beta^2 B_j{}^k{}_i y^j y_k = 0. \end{aligned}$$

Now, we assume that the Finsler space $F^n$ with $(\alpha,\beta)$-metric given by (2.1) is a Berwald space, that is, $G_j{}^i{}_k$ is a function of the position alone. Then we have $B_j{}^k{}_i = B_j{}^k{}_i(x)$, so that the second term is rational and $\alpha$ is an irrational polynomial in $(y^i)$. Thus, we have

$$\begin{aligned} 2\beta B_j{}^k{}_i y^j y_k + (c_2\beta^2 - \alpha^2)(B_j{}^r{}_i b_r - \nabla_i b_j) y^j = 0, \\ \beta^2 B_j{}^k{}_i y^j y_k = 0. \end{aligned} \tag{3.2}$$

From the above two equations, we have

$$(c_2\beta^2 - \alpha^2)(B_j{}^k{}_i b_k - \nabla_i b_j) y^j = 0. \tag{3.3}$$

Next, we proceed with different cases:

Case(a): Suppose $(c_2\beta^2 - \alpha^2) \neq 0$. Then (3.3) yields

$$(B_j{}^k{}_i b_k - \nabla_i b_j) y^j = 0,$$

which implies

$$B_j{}^k{}_i b_k - \nabla_i b_j = 0. \tag{3.4}$$

From (3.2), we have $B_j{}^k{}_i y^j y_k = 0$, which implies

$$\begin{aligned} B_j{}^k{}_i y^j y_k + B_h{}^k{}_i y^h y_k &= 0, \\ \Rightarrow B_j{}^k{}_i y^j a_{kh} y^h + B_h{}^k{}_i y^h a_{kj} y^j &= 0. \end{aligned} \tag{3.5}$$

Contracting (3.5) with $b_j b_h$, we have

$$\left(B_j{}^k{}_i a_{kh} + B_h{}^k{}_i a_{kj}\right)\beta^2 = 0,$$

which gives

$$B_j{}^k{}_i a_{kh} + B_h{}^k{}_i a_{kj} = 0, \tag{3.6}$$

From (3.6), we have $B_j{}^k{}_i = 0$ and from (3.4), we have $\nabla_i b_j = 0$.

Conversely, according to [2], if $\nabla_k b_i = 0$, then the Finsler space $F^n$ with the $(\alpha,\beta)$-metric is a Berwald space.

Case(b): Suppose $(c_2\beta^2 - \alpha^2) = 0$, which implies $c_2 = 0$. In this case, (2.1) reduces to $L = c_1\alpha + \frac{\alpha^2}{\beta}$, $c_1 \neq 0$, the special $(\alpha,\beta)$-metric. From (3.2), we can have

$$B_j{}^k{}_i y^j y_k = 0$$

and from which, we have

$$B_j{}^k{}_i a_{kh} + B_h{}^k{}_i a_{kj} = 0,$$

which gives $B_j{}^k{}_i = 0$.

On the other hand, again by [2], $F^n$ with the mentioned special $(\alpha,\beta)$-metric is a Berwald space. This completes the proof. □

Further, from (2.3) and (3.1), we get

$$(c_1\beta^2 + 2\alpha\beta)B_j{}^k{}_i y^j y_k + \alpha(c_2\beta^2 - \alpha^2)(B_j{}^k{}_i b_k - \nabla_i b_j)y^j = 0. \tag{3.7}$$

(3.7) can be rewritten as

$$(c_2\beta^2 - \alpha^2)(\nabla_i b_j)y^j = \{(c_1\beta^2 + 2\alpha\beta)e_k + (c_2\beta^2 - \alpha^2)b_k\}B^k_i, \tag{3.8}$$

where $e_k = \dfrac{y_k}{\alpha}$. We put

$$r_{ij} = \frac{(\nabla_j b_i + \nabla_i b_j)}{2}, \quad s_{ij} = \frac{(\nabla_j b_i - \nabla_i b_j)}{2}.$$

Transvecting (3.8) by $y^i$ and by using the homogeneity, we have

$$(c_2\beta^2-\alpha^2)r_{00}=2\{(c_1\beta^2+2\alpha\beta)e_k+(c_2\beta^2-\alpha^2)b_k\}B^k. \tag{3.9}$$

Conversely, differentiating (3.9) by $y^i$ and by the virtue of $\dot{\partial}_i\alpha=e_i$, $\dot{\partial}_i e_k=\dfrac{a_{ki}-e_ke_i}{\alpha}$, we have

$$\begin{aligned}(c_2\beta^2-\alpha^2)r_{i0} &+ (c_2b_i\beta-e_i\alpha)r_{00}=\{(c_1\beta^2+2\alpha\beta)e_k+(c_2\beta^2-\alpha^2)b_k\}B^k_i\\ &+ \{(c_1\beta^2+2\alpha\beta)(\frac{a_{ki}-e_ke_i}{\alpha})\\ &+ (c_1b_i\beta+b_i\alpha+e_i\beta)2e_k+(c_2b_i\beta-e_i\alpha)2b_k\}B^k.\end{aligned} \tag{3.10}$$

From (3.8),(3.9) and (3.10), we have

$$\begin{aligned}a_{ki}\left\{\frac{c_1\beta^2+2\alpha\beta}{\alpha}\right\}B^k &= (c_2b_i\beta-e_i\alpha)r_{00}+(c_2\beta^2-\alpha^2)s_{i0}\\ &\quad+\left\{\frac{c_1\beta^2+2\alpha\beta}{\alpha}\right\}e_ie_kB^k\\ &\quad-(c_1b_i\beta+b_i\alpha+e_i\beta)e_kB^k-(c_2b_i\beta-e_i\alpha)b_kB^k.\end{aligned} \tag{3.11}$$

Put $e_kB^k=E$ and $b_kB^k=D$ and divide (3.11) by $\dfrac{c_1\beta^2+2\alpha\beta}{\alpha}$, we get

$$\begin{aligned}a_{ki}B^k &= \left\{E+\frac{\alpha\left[\alpha D-(\alpha r_{00}+\beta E)\right]}{c_1\beta^2+2\alpha\beta}\right\}e_i+\frac{\alpha(c_2\beta^2-\alpha^2)}{c_1\beta^2+2\alpha\beta}s_{i0}\\ &+ \frac{\alpha\left\{c_2\beta r_{00}-\left[(c_1\beta+\alpha)E+c_2\beta D\right]\right\}}{c_1\beta^2+2\alpha\beta}b_i.\end{aligned}$$

Contract the above equation with $a^{ij}$, we obtain

$$B^i=P_1e^i+P_2s^i_0+P_3b^i, \tag{3.12}$$

where

$$\begin{aligned}P_1 &= E+\frac{\alpha\{\alpha D-(\alpha r_{00}+\beta E)\}}{c_1\beta^2+2\alpha\beta},\\ P_2 &= \frac{\alpha(c_2\beta^2-\alpha^2)}{c_1\beta^2+2\alpha\beta},\\ P_3 &= \frac{\alpha\left\{c_2\beta r_{00}-\left[(c_1\beta+\alpha)E+c_2\beta D\right]\right\}}{c_1\beta^2+2\alpha\beta}.\end{aligned} \tag{3.13}$$

From (3.9), we get

$$(c_2\beta^2-\alpha^2)r_{00}=2\{(c_1\beta^2+2\alpha\beta)E+2(c_2\beta^2-\alpha^2)D. \tag{3.14}$$

Transvecting (3.12) by $b_i$ and by virtue of $b_i e^i = \frac{\beta}{\alpha}$, we have

$$\begin{aligned} \alpha^2\beta(c_2b^2-1)r_{00}+\alpha^2(c_2\beta^2-\alpha^2)s_0 &= \left[(c_1\beta+\alpha)(b^2\alpha^2-\beta^2)\right]E \\ &+ \left[\alpha\beta(c_1\beta+\alpha+c_2b^2)\right]D. \end{aligned} \tag{3.15}$$

By solving (3.14) and (3.15), we get $D$ and $E$. Thus, we have the following:

**Theorem 3.2.** *Let $F^n$ be a Finsler space with special $(\alpha,\beta)$-metric $L(\alpha,\beta) = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}$, $c_1 \neq 0$. Then the vector field $B^i(x,y)$ in (2.2) is given by (3.12).*

**Example 1.** *In an $(\alpha,\beta)$-metric given by (2.1), if $c_1 = c_2 = 1$, then the metric*

$$L(\alpha,\beta) = \alpha + \beta + \frac{\alpha^2}{\beta} \tag{3.16}$$

*is a special $(\alpha,\beta)$-metric. For the Finsler space with the special $(\alpha,\beta)$-metric (3.16), from (3.14) and (3.15), we determine the quantities $D$ and $E$ by the following two equations.*

$$\begin{aligned} r_{00} &= \frac{2\beta(\beta+2\alpha)}{\beta^2-\alpha^2}E + 2D, \\ \alpha^2(\beta^2-\alpha^2)s_0 + \alpha^2\beta(b^2-1)r_{00} &= (\alpha+\beta)(b^2\alpha^2-\beta^2)E \\ &+ \alpha\beta(\alpha+\beta+b^2)D. \end{aligned}$$

*From the above two equations, we get*

$$\begin{aligned} D &= \frac{\{2\alpha^2\beta^2(2\alpha+\beta)(b^2-1)-(\alpha+\beta)(\beta^2-\alpha^2)(b^2\alpha^2-\beta^2)\}r_{00}+2\alpha^2\beta(2\alpha+\beta)(\beta^2-\alpha^2)s_0}{2\{\alpha\beta^2(2\alpha+\beta)(\alpha+\beta+b^2)-(\alpha+\beta)(\beta^2-\alpha^2)(b^2\alpha^2-\beta^2)\}}, \\ E &= \frac{\alpha(\beta^2-\alpha^2)\{\beta\left[3\alpha+\beta+(1-2\alpha)b^2\right]r_{00}-2\alpha(\beta^2-\alpha^2)s_0\}}{2\{\alpha\beta^2(2\alpha+\beta)(\alpha+\beta+b^2)-(\alpha+\beta)(\beta^2-\alpha^2)(b^2\alpha^2-\beta^2)\}}. \end{aligned} \tag{3.17}$$

*From (3.13),(3.14) and (3.17), we get*

$$\begin{aligned} P_1 &= \frac{A_1\alpha r_{00}+2\alpha^2\beta(\beta^2-\alpha^2)B_1s_0}{\beta(\beta+2\alpha)C}, \\ P_2 &= \frac{\alpha(\beta^2-\alpha^2)}{\beta(\beta+2\alpha)}, \\ P_3 &= \frac{A_2\alpha\beta r_{00}-2\alpha^4(\beta^2-\alpha^2)(\alpha^2+\alpha\beta+\beta^2)s_0}{\beta(\beta+2\alpha)C}, \end{aligned}$$

*where*

$$\begin{aligned} A_1 &= (\alpha+\beta)(\beta^2-\alpha^2)\left\{\beta^2(2\alpha+\beta)+b^2\left[\alpha^3+\beta^2(1-2\alpha)\right]\right\} \\ &+ 2\alpha^2\beta^2(2\alpha+\beta)\left[b^2(\alpha-1)-(2\alpha+\beta)\right], \\ B_1 &= (\alpha^2+\alpha\beta+\beta^2)-(\beta+2\alpha)(\beta^2-\alpha^2), \\ A_2 &= 2\alpha\beta^2(2\alpha+\beta)\left[(2\alpha+\beta)-b^2(\alpha-1)\right] \\ &- (\alpha+\beta)(\beta^2-\alpha^2)\left[\alpha\beta+b^2\alpha(1-\alpha)+(3\alpha^2-\beta^2)\right], \\ C &= 2\{\alpha\beta^2(2\alpha+\beta)(\alpha+\beta+b^2)-(\alpha+\beta)(\beta^2-\alpha^2)(b^2\alpha^2-\beta^2)\}. \end{aligned} \tag{3.18}$$

*Thus, in a Finsler space with the special $(\alpha,\beta)$-metric (2.1), the vector field $B^i(x,y)$ in (2.2) is given as:*

$$\begin{aligned} B^i &= \frac{\alpha\{A_1 r_{00}+2\alpha\beta(\beta^2-\alpha^2)B_1 s_0\}}{\beta(2\alpha+\beta)C}e^i+\frac{\alpha(\beta^2-\alpha^2)}{\beta(2\alpha+\beta)}s^i_0 \\ &+ \alpha\left\{\frac{A_2\beta r_{00}-2\alpha^3(\beta^2-\alpha^2)(\alpha^2+\alpha\beta+\beta^2)s_0}{\beta(2\alpha+\beta)C}\right\}b^i, \end{aligned}$$

*where $A_1$, $B_1$, $A_2$ and $C$ are as given in (3.18).*

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