Advances in Mathematics: Scientific Journal **9** (2020), no.6, 3237–3246
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.6.4

MINIMUM CLIQUE-CLIQUE DOMINATING ENERGY OF A GRAPH

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ABSTRACT. Let $C(G)$ denotes the set of all cliques of a graph G. Two cliques in G are adjacent if there is a vertex incident on them. Two cliques $c_1, c_2 \in C(G)$ are said to clique-clique dominate (cc-dominate) each other if there is a vertex incident with c_1 and c_2. A set $L \subseteq C(G)$ is said to be a cc-dominating set (CCD-set) if every clique in G is cc-dominated by some clique in L. The cc-domination number $\gamma_{cc} = \gamma_{cc}(G)$ is the order of a minimum cc-dominating set of G. In this paper we introduce minimum cc-dominating energy of the graph denoting it as $E_{cc}(G)$. It depends both on underlying graph of G and its particular minimum cc-dominating set (γ_{cc}-set) of G. Upper and lower bounds for $E_{cc}(G)$ are established.

1. INTRODUCTION

The terminologies and notations used here are as in [9, 18]. By a graph $G(V, E)$ we mean a connected finite simple graph of order p and size q. A set $D \subseteq V$ is a dominating set of G if every vertex not in D is adjacent to some vertex in D. The domination number $\gamma = \gamma(G)$ is the order of a minimum dominating set of G. The domination number is a well studied parameter in literature and for a survey refer [8, 12, 16, 17]. A maximal complete subgraph is a clique. Let C(G) denote the set of all cliques of G with $|C(G)| = s$. Two cliques in G are adjacent if there is a common vertex incident on them. A clique

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2010 *Mathematics Subject Classification.* 05C69, 05C50.

Key words and phrases. Energy of a graph, Domination number of a graph, clique-clique domination, Clique graph.

graph $C_G(G)$ is a graph with vertex set $C(G)$ and any two vertices in $C_G(G)$ are adjacent if and only if corresponding cliques are adjacent in G. The number of edges in the clique graph $C_G(G)$ is denoted as q_c. Smitha G. Bhat et al [3] defined cc-degree and cc-dominating sets as follows. The cc-degree (Clique-clique degree) of a clique h, $d_{cc}(h)$ is the number of cliques adjacent to h. Two cliques $c_1,\ c_2 \in C(G)$ are said to clique dominate each other if there is a vertex incident with c_1 and c_2. A set $L \subseteq C(G)$ is said to be a clique-clique dominating set (CCD-set) if every clique in G is clique dominated by some clique in L. The clique-clique domination number $\gamma_{cc} = \gamma_{cc}(G)$ is the order of a minimum clique-clique dominating set of G. The concept of energy of a graph was introduced by I. Gutman [4] in 1978. The eigenvalues of G are the eigenvalues of its adjacency matrix $A(G)$. These eigenvalues arranged in an non-increasing order, will be denoted as $\lambda_1(G), \lambda_2(G) \ldots, \lambda_p(G)$. Then the energy of the graph G is defined as $E(G) = \sum_{i=1}^{p} |\lambda_i(G)|$. Various properties of energy of the graph may be found in [5,6]. In connection with graph energy, energy-like quantities were considered for other matrices such as laplacian [7] distance [10], covering [1], incidence [11] and vb-dominating [15].

2. Clique-clique dominating energy of a graph

Motivated by the definition of clique-clique dominating set and energy of a graph, we introduce a new matrix, called minimum cc-dominating matrix of a graph and study its energy. Let γ_{cc}-set be a minimum cc-dominating set of a graph G. The minimum cc-dominating matrix of G is the $s \times s$ matrix $A_{cc}(G) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 \text{ if } c_i \text{ and } c_j \text{ are adjacent} \\ 1 \text{ if } i = j \text{ and } c_i \in \gamma_{cc}\text{-set} \\ 0 \text{ otherwise} . \end{cases}$$

The characteristic polynomial of $A_{cc}(G)$ is denoted by

$$f_c\left(G, \lambda\right) = det\left(\lambda I - A_{cc}(G)\right) .$$

The minimum cc-dominating eigenvalues of the graph G are the eigenvalues of $A_{cc}(G)$. Since $A_{cc}(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s$. The minimum cc-dominating energy of G is then defined as

$$E_{cc}(G) = \sum_{i=1}^{s} |\lambda_i(G)|.$$

In this paper we discuss some properties of minimum cc-dominating energy of the graph $E_{cc}(G)$ and derive an upper and lower bound for $E_{cc}(G)$.

2.1. **Properties of clique-clique dominating energy of a graph.** First we compute the minimum cc-dominating energy of the graph shown in Figure 1.

Example 1.

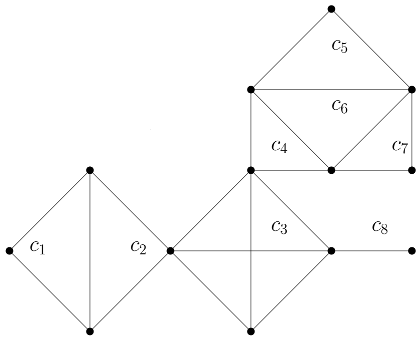

FIGURE 1. Graph G

Let G be a graph with 8 cliques $c_1, c_2, \ldots, c_8$ (See Figure 1) with minimum cc-dominating set $B = \{c_2, c_6, c_8\}$. Then

$$A_{cc}(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The characteristic polynomial of $A_{cc}(G)$ is $\lambda^8 - 3\lambda^7 - 7\lambda^6 + 15\lambda^5 + 20\lambda^4 - 13\lambda^3 - 20\lambda^2 - 5\lambda$, the minimum cc-dominating eigenvalues are $-1.7834, -1.0000, -0.7457, -0.3832, 0.0000, 1.3083, 2.2093$ and 3.3947. Therefore the minimum cc-dominating energy of the graph G is $E_{cc}(G) = 10.8246$.

Theorem 2.1. *If* $\lambda_1(G), \lambda_2(G) \ldots, \lambda_s(G)$ *are the eigenvalues of* $A_{cc}(G)$*, then*

$$\sum_{i=1}^{s} \lambda_i = \gamma_{cc}(G) \tag{2.1}$$

$$\sum_{i=1}^{s} \lambda_i^2 = 2q_c + \gamma_{cc}(G). \tag{2.2}$$

where q_c *is the number of edges in the clique graph* $C_G(G)$.

Proof. (2.1) We know that the sum of the eigenvalues of $A_{cc}(G)$ is equal to trace of $A_{cc}(G)$. Therefore

$$\sum_{i=1}^{s} \lambda_i = \sum_{i=1}^{s} a_{ii} = \gamma_{cc}(G).$$

(2.2) The sum of the squares of the eigenvalues of $A_{cc}(G)$ is the trace of $(A_{cc}(G))^2$. Therefore

$$\begin{aligned}\sum_{i=1}^{s} \lambda_i^2 &= \sum_{i=1}^{s}\sum_{j=1}^{s} (a_{ij}a_{ji}) \\ &= 2\sum_{i<j} (a_{ij})^2 + \sum_{i=1}^{s} (a_{ij})^2 \\ &= 2q_c + \gamma_{cc}(G)\end{aligned}$$

□

Theorem 2.2. *Let* G *be a graph with* q_c *edges,* s *cliques, and the minimum cc-dominating set* γ_{cc}*-set. Let* $f_s(G, \lambda) = c_0\lambda^s + c_1\lambda^{s-1} + c_2\lambda^{s-2} + \cdots + c_s$ *be the characteristic polynomial of G, Then*

$$c_0 = 1 \tag{2.3}$$

$$c_1 = -\gamma_{cc}(G) \tag{2.4}$$

$$c_2 = \binom{\gamma_{cc}(G)}{2} - q_c \tag{2.5}$$

$$c_3 = \gamma_{cc}(G)q_c - \sum_{h\in\gamma_{cc}} d_{cc}(h) - \binom{\gamma_{cc}(G)}{3} - 2\triangle \tag{2.6}$$

where q_c *and* $\triangle$ *denote the number of edges and triangles in the clique graph* $C_G(G)$ *respectively.*

Proof. (2.3) Directly follows from the definition of $f_s(G,\lambda)$.
(2.4) Since the sum of diagonal elements of $A_{cc}(G)$ is equal to $\gamma_{cc}(G)$, the sum of determinants of all 1×1 principal sub-matrices of $A_{cc}(G)$ is the trace of $A_{cc}(G)$, which is equal to $\gamma_{cc}(G)$. Thus $(-1)^1 c_1 = \gamma_{cc}(G)$.
(2.5) $(-1)^2 c_2$ is equal to the sum of determinants of all 2×2 principal sub-matrices of $A_{cc}(G)$, that is

$$\begin{aligned} c_2 &= \sum_{1\le i<j\le s} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1\le i<j\le s} (a_{ii}a_{jj} - a_{ij}a_{ji}) \\ &= \sum_{1\le i<j\le s} (a_{ii}a_{jj}) - \sum_{i\le i<j\le s} (a_{ij})^2 \\ &= \binom{\gamma_{cc}(G)}{2} - q_c\,. \end{aligned}$$

(2.6) We have

$$\begin{aligned} c_3 &= (-1)^3 \sum_{1\le i<j<k\le s} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix} \\ &= -\sum_{1\le i<j<k\le s} \left[a_{ii}\left(a_{jj}a_{kk} - a_{jk}a_{kj}\right) - a_{ij}\left(a_{ji}a_{kk} - a_{ki}a_{jk}\right) + a_{ik}\left(a_{ji}a_{kj} - a_{ki}a_{jj}\right)\right] \\ &= -\sum_{1\le i<j<k\le s} (a_{ii}a_{jj}a_{kk}) + \sum_{1\le i<j<k\le s} (a_{ii}a_{jk} + a_{jj}a_{ik} + a_{kk}a_{ij}) \\ &\quad - \sum_{1\le i<j<k\le s} (a_{ij}a_{jk}a_{ki}) - \sum_{1\le i<j<k\le s} (a_{ik}a_{kj}a_{ji}) \\ &= -\binom{\gamma_{cc}(G)}{3} + \sum_{1\le i<j<k\le s} (a_{ii}a_{jk} + a_{jj}a_{ik} + a_{kk}a_{ij}) - 2\triangle \\ &= -\binom{\gamma_{cc}(G)}{3} + \left[\sum_{i=1}^{s} a_{ii}\right]\left[\sum_{1\le i<j<k\le s} a_{jk}\right] - \sum_{i=1}^{s} a_{ii} \sum_{k=1,k\ne i} a_{ik} - 2\triangle \\ c_3 &= \gamma_{cc}(G)q_c - \sum_{h\in\gamma_{cc}(G)} d_{cc}(h) - \binom{\gamma_{cc}(G)}{3} - 2\,\triangle\,. \end{aligned}$$

□

Theorem 2.3. *If $\lambda_1(G)$ is the largest eigenvalue of the minimum cc-dominating matrix $A_{cc}(G)$, then*

$$\lambda_1(G) \geq \frac{2q_c + \gamma_{cc}(G)}{s}.$$

Proof. Let X be any non zero vector, then we have

$\lambda_1\left(A_{cc}(G)\right) = \begin{matrix} max \\ X \neq 0 \end{matrix} \left[\dfrac{X^{'}AX}{X^{'}X}\right]$ (see [2])

$\lambda_1(A_{cc}(G)) \geq \left[\dfrac{J^{'}AJ}{J^{'}J}\right] = \dfrac{2q_c + \gamma_{cc}(G)}{s}$, where J is the all one's vector. □

2.2. **Bounds for minimum cc-dominating energy of a graph.** Bounds for $E_{cc}(G)$ similar to McClelland's inequalities [14] for graph energy, are given in the following theorems.

Theorem 2.4. *Let G be the graph with s cliques, q_c edges, and minimum cc-dominating set γ_{cc}-set of G. Then*

$$E_{cc}(G) \leq \sqrt{s\left(2q_c + \gamma_{cc}(G)\right)}\,.$$

Proof. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s$ be the eigenvalues of $A_{cc}(G)$.
Using Cauchy-Schwarz inequality, $\left(\sum\limits_{i=1}^{s} a_i b_i\right)^2 \leq \left(\sum\limits_{i=1}^{s} a_i^2\right)\left(\sum\limits_{i=1}^{s} b_i^2\right)$,
choose $a_i = 1$ and $b_i = |\lambda_i|$,
$\left(E_{cc}(G)\right)^2 = \left(\sum\limits_{i=1}^{s} |\lambda_i|\right)^2 \leq \left(\sum\limits_{i=1}^{s} 1\right)\left(\sum\limits_{i=1}^{s} |\lambda_i|^2\right) = s\left(2q_c + \gamma_{cc}(G)\right)$, by using Theorem 2.1. Therefore $E_{cc}(G) \leq \sqrt{s\left(2q_c + \gamma_{cc}(G)\right)}$. □

Following theorem gives lower bound for $E_{cc}(G)$ in terms of number of cliques and number of edges in the clique graph $C_G(G)$.

Theorem 2.5. *Let G be the graph with s cliques and q_c edges, and let minimum cc-dominating set be γ_{cc}-set. If P is determinant of $A_{cc}(G)$, then*

$$E_{cc}(G) \geq \sqrt{2q_c + \gamma_{cc}(G) + s(s-1)P^{\frac{2}{s}}}\,.$$

Proof.

$$
\begin{aligned}
(E_{cc}(G))^2 &= \left(\sum_{i=1}^{s} |\lambda_i|\right)^2 \\
&= \left(\sum_{i=1}^{s} |\lambda_i|\right)\left(\sum_{j=1}^{s} |\lambda_j|\right) \\
&= \sum_{i=1}^{s} |\lambda_i|^2 + \sum_{i \neq j} |\lambda_i||\lambda_j|.
\end{aligned}
$$

Now employing the inequality between the arithmetic and geometric means, we obtain

$$
\begin{aligned}
\frac{1}{s(s-1)} \sum_{i \neq j} |\lambda_i||\lambda_j| &\geq \left(\prod_{i \neq j} |\lambda_i||\lambda_j|\right)^{\frac{1}{s(s-1)}} \\
\sum_{i \neq j} |\lambda_i||\lambda_j| &\geq s(s-1)\left(\prod_{i \neq j} |\lambda_i||\lambda_j|\right)^{\frac{1}{s(s-1)}}
\end{aligned}
$$

Thus

$$
\begin{aligned}
(E_{cc}(G))^2 &\geq \sum_{i=1}^{s} |\lambda_i|^2 + s(s-1)\left(\prod_{i \neq j} |\lambda_i||\lambda_j|\right)^{\frac{1}{s(s-1)}} \\
&\geq \sum_{i=1}^{s} |\lambda_i|^2 + s(s-1)\left(\prod_{i=1}^{s} |\lambda_i|^{2(s-1)}\right)^{\frac{1}{s(s-1)}} \\
&\geq \sum_{i=1}^{s} |\lambda_i|^2 + s(s-1)\left(\prod_{i=1}^{s} |\lambda_i|\right)^{\frac{2}{s}} \\
(E_{cc}(G))^2 &\geq 2q_c + \gamma_{cc}(G) + s(s-1)P^{\frac{2}{s}},
\end{aligned}
$$

from the Theorem 2.1. Therefore $E_{cc}(G) \geq \sqrt{2q_c + \gamma_{cc}(G) + s(s-1)P^{\frac{2}{s}}}$. □

Similar to Koolen and Moultons's [13] upper bound for energy of the graph, upper bound for $E_{cc}(G)$ is given in the following theorem.

Theorem 2.6. *If G is a graph with s cliques and q_c edges, then*

$$E_{cc}(G) \leq \frac{2q_c + \gamma_{cc}(G)}{s} + \sqrt{(s-1)\left((2q_c + \gamma_{cc}(G)) - \left(\frac{2q_c + \gamma_{cc}(G)}{s}\right)^2\right)}.$$

Proof. Using Cauchy-Schwarz inequality, $\left(\sum_{i=2}^{s} a_i b_i\right)^2 \leq \left(\sum_{i=2}^{s} a_i^2\right)\left(\sum_{i=2}^{s} b_i^2\right)$, choose $a_i = 1$ and $b_i = |\lambda_i|$

$$\begin{aligned}
\left(\sum_{i=2}^{s} |\lambda_i|\right)^2 &\leq \left(\sum_{i=2}^{s} 1\right)\left(\sum_{i=2}^{s} |\lambda_i|^2\right) \\
\left(\sum_{i=1}^{s} |\lambda_i| - \lambda_1\right)^2 &\leq (s-1)\left(\sum_{i=1}^{s} \lambda_i^2 - \lambda_1^2\right) \\
(E_{cc}(G) - \lambda_1)^2 &\leq (s-1)\left(2q_c + \gamma_{cc}(G) - \lambda_1^2\right) \\
(E_{cc}(G) - \lambda_1) &\leq \sqrt{(s-1)\left(2q_c + \gamma_{cc}(G) - \lambda_1^2\right)} \\
E_{cc}(G) &\leq \lambda_1 + \sqrt{(s-1)\left(2q_c + \gamma_{cc}(G) - \lambda_1^2\right)}.
\end{aligned}$$

Let $f(t) = t + \sqrt{(s-1)\left(2q_c + \gamma_{cc}(G) - t^2\right)}$ for decreasing function $f'(t) \leq 0$. $f'(t) = 1 - \frac{t(s-1)}{\sqrt{(s-1)\left(2q_c + \gamma_{cc}(G) - t^2\right)}} \leq 0$.

Therefore $t \geq \sqrt{\frac{2q_c + \gamma_{cc}(G)}{s}}$. Since $2q_c + \gamma_{cc} \geq s$, we have

$$\sqrt{\frac{2q_c + \gamma_{cc}}{s}} \leq \frac{2q_c + \gamma_{cc}(G)}{s} \leq \lambda_1$$

$f(\lambda_1) \leq f\left(\frac{2q_c + \gamma_{cc}}{s}\right)$. Therefore

$$E_{cc}(G) \leq f(\lambda_1) \leq f\left(\frac{2q_c + \gamma_{cc}}{s}\right)$$

$$E_{cc}(G) \leq f\left(\frac{2q_c + \gamma_{cc}(G)}{s}\right)$$

$$E_{cc}(G) \leq \frac{2q_c + \gamma_{cc}(G)}{s} + \sqrt{(s-1)\left((2q_c + \gamma_{cc}) - \left(\frac{2q_c + \gamma_{cc}}{s}\right)^2\right)}.$$

□

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