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3 - VERTEX FULL BALANCE INDEX SET OF GRAPHS

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ABSTRACT. Let G be a graph with vertex set $V(G)$ and edge set $X(G)$. Consider the set $A = \{0, 1, 2\}$. A labeling $f : V(G) \to A$ induces a partial edge labeling $f^* : X(G) \to A$ defined by $f^*(xy) = f(x)$, if and only if $f(x) = f(y)$, for each edge $xy \in X(G)$. For $i \in A$, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$ and $e_{f^*}(i) = |\{e \in X(G) : f^*(e) = i\}|$. A labeling f of a graph G is said to be 3-vertex friendly if $|v_f(i) - v_f(j)| \leq 1$, for all $i \in \{0, 1, 2\}$. The 3-vertex full balance index set of a graph G is denoted by $FBI_{3v}(G)$ and is defined as $\{e_{f^*}(i) - e_{f^*}(j),$ for $i, j = 0, 1, 2 : f^*$ runs over all 3-vertex friendly labeling f of G$\}$. In paper, we study 3-vertex full balance index set and 3-vertex balance index set of some families of graph.

1. INTRODUCTION

Let G be a graph with vertex set $V(G)$ and edge set $X(G)$. For all notations and terminologies we refer [4]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. A binary vertex labeling is said to be friendly [1], if $|v_f(1) - v_f(0)| \leq 1$, where $v_f(i)$ is the number of vertices labeled with $i = 0, 1$.

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In [7], S. M. Lee, A. Liu and S. K. Tan defined a partial edge labeling f^* of G in the following way. For each edge uv in G,

$$f^*(uv) = \begin{cases} 0, & \text{if } f(u) = f(v) = 0 \\ 1, & \text{if } f(u) = f(v) = 1. \end{cases}$$

Note that if $f(u) \neq f(v)$, then the edge uv is not labeled by f^*. In [8], A. N. T. Lee, S. M. Lee and H. K. Ng defined balance index set as $BI(G) = \{|e_{f^*}(0) - e_{f^*}(1)| : f^*$ runs over all friendly abeling f of G$\}$. W. C. Shiu and H. Kwong [5] defined full balance index set of G as $FBI(G) = \{e_{f^*}(0) - e_{f^*}(1) :$ f^* runs over all friendly labeling f of G$\}$.

A mapping $f : V(G) \rightarrow \{0, 1, 2\}$ is called ternary vertex labeling of G and $f(v)$ is called the label of vertex v of G under f, see [2]. The labeling of a graph G is said to be 3-vertex friendly labeling if $|v_f(i) - v_f(j)| \leq 1$, for all $i, j \in \{0, 1, 2\}$. More information on balance index set of graphs is obtained from the literature [3, 6, 9, 10].

With these notations, we now introduce a notion of 3-partial edge labeling, 3-vertex full balance index set and 3-vertex balance index set.

Definition 1.1. *A mapping $f^* : X(G) \rightarrow \{0, 1, 2\}$ is said to be 3-partial edge labeling if for each $uv \in X(G)$,*

$$f^*(uv) = \begin{cases} 0, & \text{if } f(u) = f(v) = 0, \\ 1, & \text{if } f(u) = f(v) = 1, \\ 2, & \text{if } f(u) = f(v) = 2. \end{cases}$$

Where $f(v)$ is called the label of vertex v of G under f. If the vertex labels of an edge are unequal, then that edge will be unlabeled by f^*. Let $v_f(i)$ and $e_{f^*}(i)$ represent the number of vertices and edges labeled with i by the mapping f and f^* respectively.

Definition 1.2. *The 3-vertex FBI set of a graph G is defined as $FBI_{3v}(G) = \{e_{f^*}(i) - e_{f^*}(j)$, for $i, j = 0, 1, 2 :$ f^* runs over all 3-vertex friendly labeling f of G$\}$.*

Definition 1.3. *The 3-vertex BI set of a graph G is defined as $BI_{3v}(G) = \{|e_{f^*}(i) - e_{f^*}(j)|$, for $i, j = 0, 1, 2 :$ f^* runs over all 3-vertex friendly labeling f of G$\}$.*

In number theory and combinatorics, a partition of a positive integer n is a method of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered to be the same partition; if order matters then the sum becomes a composition. The idea of integer partition are used to prove the theorems.

This paper is organized as follows. In Section 2, we show that 3-equitable full balance index set depends on the degree sequence of the graph and also we find the full balance index set and balance index set of some standard graphs like path, cycle, complete graph and wheel graph.

2. 3-VERTEX FULL BALANCE INDEX SET OF GRAPHS

Consider a graph with vertex labeling f. Let $e_0(x)$, $e_1(x)$ and $e_2(x)$ be total number of unlabeled edges having exactly one of the end vertex labeled $0, 1$ and 2 respectively. In this section, first we prove an algebraic approach to attempt 3-vertex full balance index set problems. It can be showed that 3-vertex full balance index set always depends on sequence of degrees of vertices.

Lemma 2.1. *For a 3-vertex friendly graph G,*

(i) $2e_{f^*}(i) + e_i(x) = \sum_{v\in V(i)} \deg(v)$, *for* $i = 0, 1, 2$.

(ii) $2|X(G)| = \sum_{i=0}^{2} \sum_{v\in V(i)} \deg(v)$,

where $e_i(x)$ is the number of unlabeled edges with one end vertex of every such edge is labeled by zero, one and two respectively.

Proof. (i) One of the vertex label of every unlabeled edge $e_i(x)$ is either $0, 1$ or 2. Also both the vertices of labeled edge has same label. Hence for a vertex v labeled i, there are exactly $\deg(v)$ edges adjacent to it. Thus

$$2e_{f^*}(i) + e_i(x) = \sum_{v\in V(i)} \deg(v), \quad for \quad i = 0, 1, 2.$$

(ii) We have $\sum_{v\in V} \deg(v) = 2|X(G)|$. For a 3-vertex friendly graph,

$$\begin{aligned}2|E(G)| &= \sum_{v\in V(0)} \deg(v) + \sum_{v\in V(1)} \deg(v) + \sum_{v\in V(2)} \deg(v) \\ &= \sum_{i=0}^{2} \sum_{v\in V(i)} \deg(v)\,.\end{aligned}$$

□

Corollary 2.1. *For any 3-vertex friendly labeling f of graph G, the 3-vertex balance index is*

$$\begin{aligned}e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v\in V(0)} \deg(v) - \sum_{v\in V(1)} \deg(v) + \alpha\right)\\ e_{f^*}(0) - e_{f^*}(2) &= \frac{1}{2}\left(\sum_{v\in V(0)} \deg(v) - \sum_{v\in V(2)} \deg(v) + \beta\right)\\ e_{f^*}(1) - e_{f^*}(2) &= \frac{1}{2}\left(\sum_{v\in V(1)} \deg(v) - \sum_{v\in V(2)} \deg(v) + \gamma\right)\end{aligned}$$

where $\alpha = e_1(x) - e_0(x),\ \beta = e_2(x) - e_0(x)$ *and* $\gamma = e_2(x) - e_1(x)$.

Here after we denote $v_f(i)$ and $e_{f^*}(i)$ as $v(i)$ and $e(i)$ respectively, for $i = 0, 1, 2$.

Theorem 2.1. *The 3-vertex FBI set of path graph P_n, $n \geq 3$ is*

$$FBI_{3v}(P_n) = \left\{ s - \left(\frac{n-r}{3}\right), \quad 1 \leq r \leq 3, \quad 0 \leq s \leq 2\left(\frac{n-r}{3}\right)\right\}.$$

Proof. Consider $n = 3k + r$, where $1 \leq r \leq 3$, $k \geq 1$.
Case 1. Let $n = 3k + 1$, $k \geq 1$. To satisfy 3-vertex friendly labeling, we partition pendant vertices and remaining vertices as given in Table 1.

Pendant vertices	Non pendant vertices	Values of k
$(i, j, 2-i-j)$	$Y_1 = \left(\frac{n-1}{3} - i + 1, \frac{n-1}{3} - j, \frac{n-7}{3} + i + j\right)$	$k = 1$, $i = 2$ or $j = 2$ or $i + j = 0$, where $i, j \in \{0, 1, 2\}$.
$(i, j, 2-i-j)$	$Y_1 = \left(\frac{n-1}{3} - i + 1, \frac{n-1}{3} - j, \frac{n-7}{3} + i + j\right)$ $Y_2 = \left(\frac{n-1}{3} - i, \frac{n-1}{3} - j + 1, \frac{n-7}{3} + i + j\right)$ $Y_3 = \left(\frac{n-1}{3} - i, \frac{n-1}{3} - j, \frac{n-4}{3} + i + j\right)$	$k \geq 1$, where $i, j \in \{0, 1, 2\}$ and $0 \leq i + j \leq 2$.

TABLE 1. Partitions of pendant and non pendant vertices

Considering the partitions displayed in Table 1 and using Corollary 2.1, we have

$$e(0) - e(1) = \frac{1}{2}\left(i + 2\left(\frac{n-1}{3} - i + 1\right) - j - 2\left(\frac{n-1}{3} - j\right) + \alpha\right)$$
$$= \frac{1}{2}\left((j - i + 2) + \alpha\right)$$
$$e(0) - e(1) = \frac{1}{2}\left(i + 2\left(\frac{n-1}{3} - i\right) - j - 2\left(\frac{n-1}{3} - j + 1\right) + \alpha\right)$$
$$= \frac{1}{2}\left((j - i - 2) + \alpha\right)$$
$$e(0) - e(1) = \frac{1}{2}\left(i + 2\left(\frac{n-1}{3} - i\right) - j - 2\left(\frac{n-1}{3} - j\right) + \alpha\right)$$
$$= \frac{1}{2}\left((j - i) + \alpha\right)$$
$$e(0) - e(2) = \frac{1}{2}\left(i + 2\left(\frac{n-1}{3} - i + 1\right) - (2 - i - j) - 2\left(\frac{n-7}{3} i + j\right) + \beta\right)$$
$$= \frac{1}{2}\left((4 - 2i - j) + \beta\right)$$
$$e(0) - e(2) = \frac{1}{2}\left(i + 2\left(\frac{n-1}{3} - i\right) - (2 - i - j) - 2\left(\frac{n-7}{3} + i + j\right) + \beta\right)$$
$$= \frac{1}{2}\left((2 - 2i - j) + \beta\right)$$
$$e(0) - e(2) = \frac{1}{2}\left(i + 2\left(\frac{n-1}{3} - i\right) - (2 - i - j) - 2\left(\frac{n-4}{3} + i + j\right) + \beta\right)$$

$$
\begin{aligned}
&= \frac{1}{2}\left((-2i-j)+\beta\right)\\
e(1)-e(2) &= \frac{1}{2}\left(j+2\left(\frac{n-1}{3}-j\right)-(2-i-j)-2\left(\frac{n-7}{3}i+j\right)+\gamma\right)\\
&= \frac{1}{2}\left((2-i-2j)+\gamma\right)\\
e(1)-e(2) &= \frac{1}{2}\left(j+2\left(\frac{n-1}{3}-j+1\right)-(2-i-j)-2\left(\frac{n-7}{3}+i+j\right)+\gamma\right)\\
&= \frac{1}{2}\left((4-i-2j)+\gamma\right)\\
e(1)-e(2) &= \frac{1}{2}\left(i+2\left(\frac{n-1}{3}-j\right)-(2-i-j)-2\left(\frac{n-4}{3}+i+j\right)+\gamma\right)\\
&= \frac{1}{2}\left((-i-2j)+\gamma\right).
\end{aligned}
$$

The evaluation of α, β and γ are shown in Table 2.
Considering all the estimated values in Table 2, we obtain

$$FBI_{3v}(P_n)=\{-k,-k+1,-k+2,\dots,k\}=\left\{\frac{1-n}{3},\frac{4-n}{3},\frac{7-n}{3},\dots,\frac{n-1}{3}\right\}.$$

Case 2. If $n=3k+2$, $k\geq 1$, then to satisfy 3-vertex friendly labeling, partition of pendant and non pendant vertices are shown in Table 3.

Pendant vertices	Non pendant vertices	Values of k
$(i,j,2-i-j)$	$Y_1=\left(\frac{n+1}{3}-i,\frac{n+1}{3}-j-1,\frac{n-5}{3}+i+j\right)$	$k=1$, $i=2$ or $j=2$ or $i+j=0$ where $i,j\in\{0,1,2\}$
$(i,j,2-i-j)$	$Y_1=\left(\frac{n+1}{3}-i,\frac{n+1}{3}-j-1,\frac{n-5}{3}+i+j\right)$ $Y_2=\left(\frac{n+1}{3}-i,\frac{n+1}{3}-j,\frac{n-8}{3}+i+j\right)$ $Y_3=\left(\frac{n+1}{3}-i-1,\frac{n+1}{3}-j,\frac{n-5}{3}+i+j\right)$	$k\geq 1$ where $i,j\in\{0,1,2\}$ and $0\leq i+j\leq 2$

TABLE 3. Partitions of pendant and non pendant vertices

Pendant	Non pen-dant	$\alpha = e_1(x) - e_0(x)$	$\beta = e_2(x) - e_0(x)$	$\gamma = e_2(x) - e_1(x)$
	Y_1	$-2k, -2k+2, \dots, 2k$	$-2k, -2k+2, \dots, 2k$	$-2k, -2k+2, \dots, 2k$
$(2,0,0)$	Y_2	$-2k+2, -2k+4, \dots, 2k+2$	$-2k+2, -2k+4, \dots, 2k$	$-2k, -2k+2, \dots, 2k$
	Y_3	$-2k+2, -2k+4, \dots, 2k+2$	$-2k+2, -2k+4, \dots, 2k+2$	$-2k, -2k+2, \dots, 2k$
	Y_1	$-2k-2, -2k, \dots, 2k-2$	$-2k, -2k+2, \dots, 2k$	$-2k, -2k+2, \dots, 2k$
$(0,2,0)$	Y_2	$-2k, -2k+2, \dots, 2k$	$-2k, -2k+2, \dots, 2k$	$-2k, -2k+2, \dots, 2k$
	Y_3	$-2k, -2k+2, \dots, 2k-2$	$-2k, -2k+2, \dots, 2k$	$-2k+2, -2k+4, \dots, 2k+2$
	Y_1	$-2k, -2k+2, \dots, 2k$	$-2k-2, -2k, \dots, 2k-2$	$-2k, -2k+2, \dots, 2k-2$
$(0,0,2)$	Y_2	$-2k, -2k+2, \dots, 2k$	$-2k, -2k+2, \dots, 2k-2$	$-2k-2, -2k+2, \dots, 2k-2$
	Y_3	$-2k, -2k+2, \dots, 2k$	$-2k, -2k+2, \dots, 2k$	$-2k, -2k+2, \dots, 2k$
	Y_1	$-2k-2, -2k, \dots, 2k$	$-2k-1, -2k+1, \dots, 2k+1$	$-2k+1, -2k+3, \dots, 2k+1$
$(1,1,0)$	Y_2	$-2k, -2k+2, \dots, 2k+2$	$-2k+1, -2k+3, \dots, 2k+1$	$-2k-1, -2k+1, \dots, 2k+1$
	Y_3	$-2k, -2k+2, \dots, 2k$	$-2k+1, -2k+3, \dots, 2k+1$	$-2k+1, -2k+3, \dots, 2k+1$
	Y_1	$-2k-1, -2k+1, \dots, 2k+1$	$-2k, -2k+2, \dots, 2k$	$-2k-1, -2k+1, \dots, 2k-1$
$(1,0,1)$	Y_2	$-2k+1, -2k+3, \dots, 2k+1$	$-2k, -2k+2, \dots, 2k$	$-2k-1, -2k+1, \dots, 2k-1$
	Y_3	$-2k+1, -2k+3, \dots, 2k+1$	$-2k, -2k+2, \dots, 2k+2$	$-2k-1, -2k+1, \dots, 2k+1$
	Y_1	$-2k-1, -2k+1, \dots, 2k-1$	$-2k-1, -2k+1, \dots, 2k-1$	$-2k, -2k+2, \dots, 2k$
$(0,1,1)$	Y_2	$-2k-1, -2k+1, \dots, 2k-1$	$-2k-1, -2k+1, \dots, 2k-1$	$-2k-2, -2k, \dots, 2k$
	Y_3	$-2k-1, -2k+1, \dots, 2k-1$	$-2k-1, -2k+1, \dots, 2k+1$	$-2k, -2k+2, \dots, 2k+2$

TABLE 2. Estimation of α, β and γ for partition of pendant and non pendant vertices

Pendant	Non pendant	$\alpha = e_1(x) - e_0(x)$	$\beta = e_2(x) - e_0(x)$	$\gamma = e_2(x) - e_1(x)$
	Y_1	$-2k+2, -2k+4, \ldots, 2k-2$	$-2k+2, -2k+4, \ldots, 2k$	$-2k+2, -2k+4, \ldots, 2k$
$(2,0,0)$	Y_2	$-2k+2, -2k+4, \ldots, 2k$	$-2k+2, -2k+4, \ldots, 2k-2$	$-2k, -2k+2, \ldots, 2k-2$
	Y_3	$-2k+4, -2k+6, \ldots, 2k$	$-2k+4, -2k+6, \ldots, 2k$	$-2k+2, -2k+4, \ldots, 2k-2$
	Y_1	$-2k, -2k+2, \ldots, 2k-4$	$-2k+2, -2k+4, \ldots, 2k-2$	$-2k+4, -2k+6, \ldots, 2k$
$(0,2,0)$	Y_2	$-2k, -2k+2, \ldots, 2k-2$	$-2k, -2k+2, \ldots, 2k-2$	$-2k+2, -2k+4, \ldots, 2k-2$
	Y_3	$-2k+2, -2k+4, - \ldots, 2k-2$	$-2k+2, -2k+4, \ldots, 2k$	$-2k+2, -2k+4, \ldots, 2k$
	Y_1	$-2k, -2k+2, \ldots, 2k$	$-2k, -2k+2, \ldots, 2k-4$	$-2k+2, -2k+4, \ldots, 2k-2$
$(0,0,2)$	Y_2	$-2k+2, -2k+4, \ldots, 2k-2$	$-2k, -2k+2, \ldots, 2k-4$	$-2k, -2k+2, \ldots, 2k-4$
	Y_3	$-2k+2, -2k+4, \ldots, 2k$	$-2k+2, -2k+4, - \ldots, 2k-2$	$-2k, -2k+2, \ldots, 2k-2$
	Y_1	$-2k, -2k+2, \ldots, 2k-2$	$-2k+1, -2k+3, \ldots, 2k-1$	$-2k+3, -2k+5, \ldots, 2k+1$
$(1,1,0)$	Y_2	$-2k, -2k+2, \ldots, 2k$	$-2k+1, -2k+3, \ldots, 2k-1$	$-2k+1, -2k+3, \ldots, 2k-1$
	Y_3	$-2k+2, -2k+4, \ldots, 2k$	$-2k+3, -2k+5, \ldots, 2k+1$	$-2k+1, -2k+3, \ldots, 2k-1$
	Y_1	$-2k+1, -2k+3, \ldots, 2k-1$	$-2k, -2k+2, \ldots, 2k$	$-2k+1, -2k+3, \ldots, 2k-1$
$(1,0,1)$	Y_2	$-2k+1, -2k+3, \ldots, 2k-1$	$-2k, -2k+2, \ldots, 2k-2$	$-2k-1, -2k+1, \ldots, 2k-3$
	Y_3	$-2k+3, -2k+5, \ldots, 2k+1$	$-2k+2, -2k+4, \ldots, 2k$	$-2k+1, -2k+3, \ldots, 2k-1$
	Y_1	$-2k+1, -2k+3, \ldots, 2k-1$	$-2k-1, -2k+1, \ldots, 2k-3$	$-2k, -2k+2, \ldots, 2k-2$
$(0,1,1)$	Y_2	$-2k+1, -2k+3, \ldots, 2k-1$	$-2k-1, -2k+1, \ldots, 2k-3$	$-2k, -2k+2, \ldots, 2k-2$
	Y_3	$-2k+1, -2k+3, \ldots, 2k-1$	$-2k+1, -2k+3, \ldots, 2k-1$	$-2k, -2k+2, \ldots, 2k$

TABLE 4. Estimation of α, β and γ for partition of pendant and non pendant vertices

Proceeding in similar lines of Case 1, we obtain

$$FBI_{3v}(P_n) = \{-k, -k+1, -k+2, \ldots, k\} = \left\{\frac{2-n}{3}, \frac{5-n}{3}, \frac{8-n}{3}, \ldots, \frac{n-2}{3}\right\}.$$

Case 3. If $n = 3k+3$, $k \geq 0$, then to satisfy 3-vertex friendly labeling, pendant vertices and remaining vertices of degree two are partitioned into $(i, j, 2-i-j)$ and $(\frac{n}{3}-i, \frac{n}{3}-j, \frac{n-6}{3}+i+j)$ respectively, where $i, j \in \{0,1,2\}$ and $0 \leq i+j \leq 2$. Therefore, by Corollary 2.1,

$$\begin{aligned}
e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(i + 2\left(\frac{n}{3} - i\right) - j - 2\left(\frac{n}{3} - j\right) + \alpha\right) \\
&= \frac{1}{2}(j - i + \alpha) \\
e_{f^*}(0) - e_{f^*}(2) &= \frac{1}{2}\left(i + 2\left(\frac{n}{3} - i\right) - (2 - i - j) - 2\left(\frac{n-6}{3} + i + j\right) + \beta\right) \\
&= \frac{1}{2}(2 - j - 2i + \beta) \\
e_{f^*}(1) - e_{f^*}(2) &= \frac{1}{2}\left(j + 2\left(\frac{n}{3} - j\right) - (2 - i - j) - 2\left(\frac{n-6}{3} + i + j\right) + \gamma\right) \\
&= \frac{1}{2}(2 - 2j - i + \gamma).
\end{aligned}$$

The computation of α, β and γ are shown in Table 5.

Pendant	$\alpha = e_1(x) - e_0(x)$	$\beta = e_2(x) - e_0(x)$	$\gamma = e_2(x) - e_1(x)$
$(0,2,0)$	$-2k, -2k+2, \ldots, 2k-2$	$-2k, -2k+2, \ldots, 2k$	$-2k+2, -2k+4, \ldots, 2k$
$(0,0,2)$	$-2k, -2k+2, \ldots, 2k$	$-2k, -2k+2, \ldots, 2k-2$	$-2k, -2k+2, \ldots, 2k-2$
$(1,1,0)$	$-2k, -2k+2, \ldots, 2k$	$-2k+1, -2k+3, \ldots, 2k+1$	$-2k+1, -2k+3, \ldots, 2k+1$
$(1,0,1)$	$-2k+1, -2k+3, \ldots, 2k+1$	$-2k, -2k+2, \ldots, 2k$	$-2k-1, -2k+1, \ldots, 2k-1$
$(0,1,1)$	$-2k-1, -2k+1, \ldots, 2k-1$	$-2k-1, -2k+3, \ldots, 2k-1$	$-2k, -2k+2, \ldots, 2k$

TABLE 5. Estimation of α, β and γ for partitions of pendant vertices

By considering all possibilities of α, β and γ of Table 5, full balance index of P_n is given as

$$FBI_{3v}(P_n) = \{-k, -k{+}1, -k{+}2, \ldots, k\}, = \left\{\frac{3-n}{3}, \frac{6-n}{3}, \frac{9-n}{3}, \ldots, \frac{n-3}{3}\right\}.$$

Combining all the above three cases, we obtain

$$FBI_{3v}(P_n) = \left\{s - \left(\frac{n-r}{3}\right), \quad 1 \leq r \leq 3, \quad 0 \leq s \leq 2\left(\frac{n-r}{3}\right)\right\}. \qquad \square$$

Corollary 2.2. *If $n = 3k + r$, for $1 \leq r \leq 3$, $k \geq 0$, then the 3-vertex balance index set of path graph P_n, $n \geq 3$ is $BI_{3v}(P_n) = \left\{0, 1, 2, \ldots, \frac{n-r}{3}\right\}$.*

Theorem 2.2. *The 3-vertex full balance index set of cycle graph C_n is*

$$FBI_{3v}(C_n) = \left\{s - \left(\frac{n-r}{3}\right), \quad 1 \leq r \leq 3, \quad 0 \leq s \leq 2\left(\frac{n-r}{3}\right)\right\}.$$

Proof. Consider $n = 3k + r$, where $1 \leq r \leq 3$, $k \geq 1$.
Case 1. If $n = 3k + 1$, $k \geq 1$, then to satisfy 3-vertex friendly labeling, we must divide as n vertices depicted in Table 6.

n vertices	$\alpha = e_1(x) - e_0(x)$	$\beta = e_2(x) - e_0(x)$	$\gamma = e_2(x) - e_1(x)$
$\left(\frac{n-1}{3}, \frac{n-1}{3}, \frac{n+2}{3}\right)$	$-2k+2, -2k+4, \ldots, 2k-2$	$-2k+2, -2k+4, \ldots, 2k$	$-2k+2, -2k+4, \ldots, 2k$
$\left(\frac{n-1}{3}, \frac{n+2}{3}, \frac{n-1}{3}\right)$	$-2k+2, -2k+4, \ldots, 2k$	$-2k+2, -2k+4, \ldots, 2k-2$	$-2k, -2k+2, \ldots, 2k-2$
$\left(\frac{n+2}{3}, \frac{n-1}{3}, \frac{n-1}{3}\right)$	$-2k, -2k+2, \ldots, 2k-2$	$-2k, -2k+2, \ldots, 2k-2$	$-2k+2, -2k+4, \ldots, 2k-2$

TABLE 6. Estimation of α, β and γ for partition of n vertices

Considering the partitions in Table 6 and using Corollary 2.1, we have

$$FBI_{3v}(C_n) = \{-k, -k+1, -k+2, \ldots, k\} = \left\{\frac{1-n}{3}, \frac{4-n}{3}, \frac{7-n}{3}, \ldots, \frac{n-1}{3}\right\}.$$

Case 2. If $n = 3k + 2$, $k \geq 1$, then to satisfy 3-vertex friendly labeling form a partition as per Table 7.

n vertices	$\alpha = e_1(x) - e_0(x)$	$\beta = e_2(x) - e_0(x)$	$\gamma = e_2(x) - e_1(x)$
$\left(\frac{n-2}{3}, \frac{n+1}{3}, \frac{n+1}{3}\right)$	$-2k+2, -2k+4, \ldots, 2k$	$-2k+2, -2k+4, \ldots, 2k$	$-2k+2, -2k+4, \ldots, 2k$
$\left(\frac{n+1}{3}, \frac{n-2}{3}, \frac{n+1}{3}\right)$	$-2k, -2k+2, \ldots, 2k-2$	$-2k, -2k+2, \ldots, 2k$	$-2k+2, -2k+4, \ldots, 2k$
$\left(\frac{n+1}{3}, \frac{n+1}{3}, \frac{n-2}{3}\right)$	$-2k, -2k+2, \ldots, 2k$	$-2k, -2k+2, \ldots, 2k-2$	$-2k, -2k+2, \ldots, 2k-2$

TABLE 7. Estimation of α, β and γ for partition of n vertices

Considering the partition in Table 7 and using Corollary 2.1, we have

$$FBI_{3v}(C_n) = \{-k, -k+1, -k+2, \ldots, k\} = \left\{\frac{2-n}{3}, \frac{5-n}{3}, \frac{9-n}{3}, \ldots, \frac{n-2}{3}\right\}.$$

Case 3. If $n = 3k+3$, $k \geq 0$, then to fulfill 3-vertex friendly labeling, n vertices are partitioned into $\left(\frac{n}{3}, \frac{n}{3}, \frac{n}{3}\right)$.

Therefore, $\alpha = e_1(x) - e_0(x) \in \{-2k, -2k+2, -2k+4, \ldots, 2k\}$, $\beta = e_2(x) - e_0(x) \in \{-2k, -2k+2, -2k+4, \ldots, 2k\}$ and $\gamma = e_2(x) - e_1(x) \in \{-2k, -2k+2, -2k+4, \ldots, 2k\}$.

Hence, by Corollary 2.1, we get

$$FBI_{3v}(C_n) = \{-k, -k+1, -k+2, \ldots, k\} = \left\{\frac{3-n}{3}, \frac{6-n}{3}, \frac{9-n}{3}, \ldots, \frac{n-3}{3}\right\}.$$

Combining all the above three cases, we obtain

$$FBI_{3v}(C_n) = \left\{s - \left(\frac{n-r}{3}\right), \quad 1 \leq r \leq 3, \quad 0 \leq s \leq 2\left(\frac{n-r}{3}\right)\right\}.$$

□

Corollary 2.3. *If $n = 3k+r$, for $1 \leq r \leq 3$, $k \geq 0$, then the 3-vertex balance index set of cycle graph C_n is $BI_{3v}(C_n) = \left\{0, 1, 2, \ldots, \frac{n-r}{3}\right\}$.*

Theorem 2.3. *The 3-vertex full balance index set of K_n, $n \geq 3$ is*

$$FBI_{3v}(K_n) = \begin{cases} \{0\} & if\ n \equiv 0(\mod 3), \\ \left\{\frac{1-n}{3}, 0, \frac{n-1}{3}\right\} & if\ n \equiv 1(\mod 3), \\ \left\{\frac{2-n}{3}, 0, \frac{n-2}{3}\right\} & if\ n \equiv 2(\mod 3). \end{cases}$$

Proof. For 3-vertex friendly labeling,

Case 1. If $n = 3k+3$, $k \geq 0$ or $n \equiv 0(\mod 3)$, then to fulfill 3-vertex friendly labeling, n vertices are partitioned into $\left(\frac{n}{3}, \frac{n}{3}, \frac{n}{3}\right)$. Each of $\frac{n}{3}$ vertices are labeled zero, one and two respectively, forms a complete subgraph of K_n. Therefore, $e(i) = {}^{\frac{n}{3}}C_2$, where $i = 0, 1, 2$. Thus, by Definition 1.2, $FBI_{3v}(K_n) = \{0\}$.

Case 2. If $n = 3k+1$, $k \geq 1$ or $n \equiv 1(\mod 3)$, then to fulfill 3-vertex friendly labeling, n vertices are partitioned into $Y_1 = \left(\frac{n-1}{3}, \frac{n-1}{3}, \frac{n+2}{3}\right)$, $Y_2 = \left(\frac{n-1}{3}, \frac{n+2}{3}, \frac{n-1}{3}\right)$ and $Y_3 = \left(\frac{n+2}{3}, \frac{n-1}{3}, \frac{n-1}{3}\right)$ respectively. Thus each component of partition forms a complete subgraph of K_n. For Y_1, $e(0) = e(1) = {}^{\frac{n-1}{3}}C_2 = \frac{(n-1)(n-4)}{18}$ and $e(2) = {}^{\frac{n+2}{3}}C_2 = \frac{(n-1)(n+2)}{18}$. Similarly for Y_2, $e(0) = e(2) = \frac{(n-1)(n-4)}{18}$, $e(1) = \frac{(n-1)(n+2)}{18}$ and for Y_3, $e(0) = \frac{(n+2)(n-1)}{18}$, $e(1) = e(2) = \frac{(n-1)(n-4)}{18}$. Thus, by Definition 1.2, $FBI_{3v}(K_n) = \left\{\frac{1-n}{3}, 0, \frac{n-1}{3}\right\}$.

Case 3. If $n = 3k+2$, $k \geq 1$ or $n \equiv 2(\mod 3)$, then to fulfill 3-vertex friendly labeling, n vertices are partitioned into $Y_1 = \left(\frac{n-2}{3}, \frac{n+1}{3}, \frac{n+1}{3}\right)$, $Y_2 = \left(\frac{n+1}{3}, \frac{n-2}{3}, \frac{n+1}{3}\right)$ and $Y_3 = \left(\frac{n+1}{3}, \frac{n+1}{3}, \frac{n-2}{3}\right)$ respectively. Proceeding in similar lines of Case 2, $FBI_{3v}(K_n) = \left\{\frac{2-n}{3}, 0, \frac{n-2}{3}\right\}$. □

Corollary 2.4. *The 3-vertex balance index set of K_n, $n \geq 3$ is*

$$BI_{3v}(K_n) = \begin{cases} \{0\} & if\ n \equiv 0(\mod 3), \\ \left\{0, \frac{n-1}{3}\right\} & if\ n \equiv 1(\mod 3), \\ \left\{0, \frac{n-2}{3}\right\} & if\ n \equiv 2(\mod 3). \end{cases}$$

Theorem 2.4. *If $n \geq 4$, then 3-vertex full balance index of wheel graph W_n is*

$$FBI_{3v}(W_n) = \left\{1 + s - 2\left(\frac{n-r}{3}\right), \quad 1 \leq r \leq 3, \quad 0 \leq s \leq 4\left(\frac{n-r}{3}\right) - 2\right\}.$$

Proof. Consider $n = 3k+r$, where $1 \leq r \leq 3$, $k \geq 1$.

Case 1. $n = 3k+1$, $k \geq 1$.

If $k = 1$, then $W_4 \cong K_4$. Hence the result follows from Theorem 2.3.
For $k > 1$, to satisfy 3-vertex friendly labeling, partition of central vertex and remaining vertices are given in Table 8.

Central vertex	Degree three vertices
$(i, j, 1-i-j)$ where $i, j \in \{0, 1\}$ and $0 \le i+j \le 1$	$Y_1 = \left(\frac{n-1}{3} - i, \frac{n-1}{3} - j, \frac{n-1}{3} + i + j\right)$ $Y_2 = \left(\frac{n-1}{3} - i, \frac{n-1}{3} - j + 1, \frac{n-4}{3} + i + j\right)$ $Y_3 = \left(\frac{n-1}{3} - i + 1, \frac{n-1}{3} - j, \frac{n-4}{3} + i + j\right)$

TABLE 8. Partition of central vertex and degree three vertices

Considering the partitions displayed in Table 8 and using the Corollary 2.1, we have

$$
\begin{aligned}
e(0) - e(1) &= \frac{1}{2}\left((n-1)i + 3\left(\frac{n-1}{3} - i\right) - (n-1)j - 3\left(\frac{n-1}{3} - j\right) + \alpha\right) \\
&= \frac{1}{2}((n-4)(i-j) + \alpha) \\
e(0) - e(1) &= \frac{1}{2}\left((n-1)i + 3\left(\frac{n-1}{3} - i\right) - (n-1)j - 3\left(\frac{n-1}{3} - j + 1\right) + \alpha\right) \\
&= \frac{1}{2}((n-4)(i-j) - 3 + \alpha) \\
e(0) - e(1) &= \frac{1}{2}\left((n-1)i + 3\left(\frac{n-1}{3} - i + 1\right) - (n-1)j - 3\left(\frac{n-1}{3} - j\right) + \alpha\right) \\
&= \frac{1}{2}((n-4)(i-j) + 3 + \alpha) \\
e(0) - e(2) &= \frac{1}{2}\left((n-1)i + 3\left(\frac{n-1}{3} - i\right) - (n-1)(1-i-j) - 3\left(\frac{n-1}{3} + i + j\right) + \beta\right) \\
&= \frac{1}{2}((2i+j)(n-4) - (n-1) + \beta) \\
e(0) - e(2) &= \frac{1}{2}\left((n-1)i + 3\left(\frac{n-1}{3} - i\right) - (n-1)(1-i-j) - 3\left(\frac{n-4}{3} + i + j\right) + \beta\right) \\
&= \frac{1}{2}((2i+j-1)(n-4) + \beta) \\
e(0) - e(2) &= \frac{1}{2}\left((n-1)i + 3\left(\frac{n-1}{3} - i + 1\right) - (n-1)(1-i-j) - 3\left(\frac{n-4}{3} + i + j\right) + \beta\right) \\
&= \frac{1}{2}((2i+j)(n-4) - (n-7) + \beta)
\end{aligned}
$$

$$e(1) - e(2) = \frac{1}{2}\left((n-1)j + 3\left(\frac{n-1}{3} - j\right) - (n-1)(1-i-j) - 3\left(\frac{n-1}{3} + i + j\right) + \gamma\right)$$
$$= \frac{1}{2}((2j+i)(n-4) - (n-1) + \gamma)$$
$$e(1) - e(2) = \frac{1}{2}\left((n-1)j + 3\left(\frac{n-1}{3} - j + 1\right) - (n-1)(1-i-j) - 3\left(\frac{n-4}{3} + i + j\right) + \gamma\right)$$
$$= \frac{1}{2}((2j+i)(n-4) - (n-7) + \gamma)$$
$$e(1) - e(2) = \frac{1}{2}\left((n-1)j + 3\left(\frac{n-1}{3} - j\right) - (n-1)(1-i-j) - 3\left(\frac{n-4}{3} + i + j\right) + \gamma\right)$$
$$= \frac{1}{2}((2j+i-1)(n-4) + \gamma)$$

The computation of α, β and γ are shown in Table 9,

Central	Degree 3	$\alpha = e_1(x) - e_0(x)$	$\beta = e_2(x) - e_0(x)$	$\gamma = e_2(x) - e_1(x)$
$(1,0,0)$	Y_1	$-3k+3, -3k+5, \ldots, k-3$	$-3k+4, -3k+6, \ldots, k$	$-2k+3, -2k+5, \ldots, 2k-1$
	Y_2	$-3k+4, -3k+6, \ldots, k$	$-3k+3, -3k+1, \ldots, k-3$	$-2k+1, -2k+3, \ldots, 2k-3$
	Y_3	$-3k+2, -3k+4, \ldots, k-2$	$-3k+2, -3k+4, \ldots, k-2$	$-2k+2, -2k+4, \ldots, 2k-2$
$(0,1,0)$	Y_1	$-k+3, -k+5, \ldots, 3k-3$	$-2k+3, -2k+5, \ldots, 2k-1$	$-3k+4, -3k+6, \ldots, k$
	Y_2	$-k+2-k+4, \ldots, 3k-2$	$-2k+2, -2k+4, \ldots, 2k-2$	$-3k+2, -3k+4, \ldots, k-2$
	Y_3	$-k, -k+2, \ldots, 3k-4$	$-2k+1, -2k+3, \ldots, 2k-3$	$-3k+3, -3k+5, \ldots, k-3$
$(0,0,1)$	Y_1	$-2k+2, -2k+4, \ldots, 2k-2$	$-k+2, -k+4, \ldots, 3k-2$	$-k+2, -k+4, \ldots, 3k-2$
	Y_2	$-2k+3, -2k+5, \ldots, 2k-1$	$-k+3, -k+5, \ldots, 3k-3$	$-k, -k+2, \ldots, 3k-4$
	Y_3	$-2k+1, -2k+3, \ldots, 2k-3$	$-k, -k+2, \ldots, 3k-4$	$-k+3, -k+5, \ldots, 3k-3$

TABLE 9. Estimation of α, β and γ for partition of central vertex and degree three vertices

Considering all the estimated values of α, β and γ, we obtain

$$FBI_{3v}(W_n) = \{-2k+1, -2k+2, \ldots, 2k-1\} = \left\{\frac{-2n+5}{3}, \frac{-2n+8}{3}, \ldots, \frac{2n-5}{3}\right\}.$$

Case 2. If $n = 3k+2$, $k \geq 1$, then to satisfy 3-vertex friendly labeling, partition of central vertex and remaining vertices are given in Table 10.

Central vertex	Degree three vertices
$(i, j, 1-i-j)$ where $i, j \in \{0,1\}$ and $0 \leq i+j \leq 1$.	$Y_1 = \left(\frac{n+1}{3} - i, \frac{n+1}{3} - j - 1, \frac{n-2}{3} + i + j\right)$ $Y_2 = \left(\frac{n+1}{3} - i, \frac{n+1}{3} - j, \frac{n-5}{3} + i + j\right)$ $Y_3 = \left(\frac{n+1}{3} - i - 1, \frac{n+1}{3} - j, \frac{n-2}{3} + i + j\right)$

TABLE 10. Partitions of central vertex and degree three vertices

Using all possible partitions given in Table 10, we get the values of α, β, γ as follows.

Except for $n = 5$, in Table 11, partitions { $(1,0,0)$ and Y_3}, {$(0,1,0)$ and Y_1}, and {$(0,0,1)$ and Y_2}, give $\alpha = \beta = 0, 2$, $\alpha = -\gamma = 0, 2$ and $\alpha = \beta = -2, 0$ respectively.

Proceeding in similar lines of Case 1, we obtain

$$FBI_{3v}(W_n) = \{-k, -k+1, -k+2, \ldots, k\} = \left\{\frac{-2n+7}{3}, -\frac{-2n+10}{3}, \ldots, \frac{2n-7}{3}\right\}.$$

Case 3. If $n = 3k+3$, $k \geq 1$, then to satisfy 3-vertex friendly labeling, central vertex and remaining vertices are partitioned into $(i, j, 1-i-j)$ and $\left(\frac{n}{3} - i, \frac{n}{3} - j, \frac{n-3}{3} + i + j\right)$ respectively, where $i, j \in \{0,1\}$ and $0 \leq i+j \leq 1$. Hence, by Corollary 2.1, we obtain

$$\begin{aligned}
e(0) - e(1) &= \frac{1}{2}\left((n-1)i + 3\left(\frac{n}{3} - i\right) - (n-1)j - 3\left(\frac{n}{3} - j\right) + \alpha\right) \\
&= \frac{1}{2}\left((n-4)(i-j) + \alpha\right) \\
e(0) - e(2) &= \frac{1}{2}\left((n-1)i + 3\left(\frac{n}{3} - i\right) - (n-1)(1-i-j) - 3\left(\frac{n-3}{3} + i + j\right) + \beta\right) \\
&= \frac{1}{2}\left((n-4)(2i+j-1) + \beta\right)
\end{aligned}$$

Central	Degree 3	$\alpha = e_1(x) - e_0(x)$	$\beta = e_2(x) - e_0(x)$	$\gamma = e_2(x) - e_1(x)$
$(1,0,0)$	Y_1	$-3k+1, -3k+3, \dots, k-3$	$-3k+2, -3k+4, \dots, k$	$-2k+3, -2k+5, \dots, 2k+1$
	Y_2	$-3k+2, -3k+4, \dots, k$	$-3k+1, -3k+3, \dots, k-3$	$-2k+2, -2k+4, \dots, 2k-2$
	Y_3	$-3k+3, -3k+5, \dots, k-1$	$-3k+3, -3k+5, \dots, k-1$	$-2k+2, -2k+4, \dots, 2k-2$
$(0,1,0)$	Y_1	$-k+1, -k+3, \dots, 3k-3$	$-2k+2, -2k+4, \dots, 2k-2$	$-3k+3, -3k+5, \dots, k-1$
	Y_2	$-k-k+2, \dots, 3k-2$	$-2k-1, -2k+1, \dots, 2k-3$	$-3k+1, -3k+3, \dots, k-3$
	Y_3	$-k+3, -k+5, \dots, 3k-1$	$-2k+3, -2k+5, \dots, 2k+1$	$-3k+1, -3k+3, \dots, k$
$(0,0,1)$	Y_1	$-2k-1, -2k+1, \dots, 2k-3$	$-k, -k+2, \dots, 3k-2$	$-k+3, -k+5, \dots, 3k-1$
	Y_2	$-2k+2, -2k+4, \dots, 2k-2$	$-k+1, -k+3, \dots, 3k-3$	$-k+1, -k+3, \dots, 3k-3$
	Y_3	$-2k+3, -2k+5, \dots, 2k+1$	$-k+3, -k+5, \dots, 3k-1$	$-k, -k+2, \dots, 3k-2$

TABLE 11. Estimation of α, β and γ for partition of central vertex and degree three vertices

$$\begin{aligned} e(1) - e(2) &= \frac{1}{2}\left((n-1)j + 3\left(\frac{n}{3} - j\right) - (n-1)(1-i-j) - 3\left(\frac{n-3}{3} + i + j\right) + \gamma\right) \\ &= \frac{1}{2}\left((n-4)(2j+i-1) + \gamma\right). \end{aligned}$$

Computation of α, β and γ are given in Table 12.
Considering all the above estimated values, we obtain

$$FBI_{3v}(W_n) = \{-2k+1, -2k+2, \dots, 2k-1\} = \left\{\frac{-2n+9}{3}, \frac{-2n+12}{3}, \dots, \frac{2n-9}{3}\right\}.$$

Combining all the above three cases, we obtain

$$FBI_{3v}(W_n) = \left\{1 + s - 2\left(\frac{n-r}{3}\right), \quad 1 \le r \le 3, \quad 0 \le s \le 4\left(\frac{n-r}{3}\right) - 2\right\}.$$

□

Central vertex	$\alpha = e_1(x) - e_0(x)$	$\beta = e_2(x) - e_0(x)$	$\gamma = e_2(x) - e_1(x)$
$(1,0,0)$	$-3k+1, -3k+3, \ldots, k-1$	$-3k+1, -3k+3, \ldots, k-1$	$-2k, -2k+2, \ldots, 2k$
$(0,1,0)$	$-k+1, -k+3, \ldots, 3k-1$	$-2k, -2k+2, \ldots, 2k$	$-3k+1, -3k+3, \ldots, k-1$
$(0,0,1)$	$-2k, -2k+2, \ldots, 2k$	$-k+1, -k+3, \ldots, 3k-1$	$-k+1, -k+3, \ldots, 3k-1$

TABLE 12. Estimation of α, β and γ for partitions of central vertex

Corollary 2.5. *If $n = 3k + r$, for $1 \leq r \leq 3$, $k \geq 1$, then the 3-vertex balance index set of wheel graph W_n, $n \geq 4$ is* $BI_{3v}(W_n) = \left\{0, 1, 2, \ldots, 2\left(\frac{n-r}{3}\right) - 1\right\}$.

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