ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.6, 3265–3270
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.6.6

COMPLETE COTOTAL EDGE DOMINATION NUMBER OF CERTAIN GRAPHS

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ABSTRACT. A total edge dominating set D is said to be a complete cototal edge dominating set if $\langle E - D\rangle$ has without isolated edges and it is represented by $\gamma^{'}_{cctd}(G)$. The complete cototal edge domination number, represented by $\gamma^{'}_{cc}(G)$, is the minimum cardinality of a complete cototal edge dominating set of G . The main purpose of this paper is to investigate the complete cototal edge domination number of certain graphs and its bounds.

1. INTRODUCTION

Domination theory in graph was developed by Claude Berge around 1960's with the problem of placing minimum number of queens on a $n \times n$ chess board to dominate each square by at least one queen. After that Oystein Ore developed the concept dominating set and domination number [6]. A set S of nodes of G is a dominating set of G if each node of G is dominated by some node in S. Cockayne, Dawes and Hedetniemi was presented by the total domination in graphs [3]. Mitchell and Hedetniemi was presented by the concept of edge domination [4]. A subset D of E is called an edge dominating set of G if every edge not in D is adjacent to some edge in D. A total edge dominating set for a graph G is a edge dominating set M for G with the properly that every edge in

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2010 *Mathematics Subject Classification.* 05C69.

Key words and phrases. Total dominating set, Total edge dominating set, Cototal edge dominating set, Cototal edge domination number, Complete cototal edge dominating set, Complete cototal edge domination number.

M has a neighbor in M and it is denoted by γ'_{td}, [2]. Note that total edge dominating sets are not defined for graphs with isolated edges. Kulli, Janakiram and Iyer was presented by the concept of cototal dominating set [5]. A dominating set D is said to be a cototal edge dominating set if $\langle E - D \rangle$ has without isolated edges. The cototal edge domination number of G is the minimum cardinality of a cototal edge dominating set of G and it is represented by $\gamma'_{ctd}(G)$ [1]. This concept motivate us to do research under this topic.

Throughout this paper we considered a simple connected graph. Let $G = (V(G), E(G))$ where $V(G)$ represents the node set and $E(G)$ represents the edge set. The total number of nodes and edges are represented by p and q respectively. An edge is said to be an isolated edge if both of its node has degree one.

2. Definition

Definition 2.1. *A total edge dominating set D is said to be a complete cototal edge dominating set if the induced subgraph $\langle E - D \rangle$ has without isolated edges. The complete cototal edge domination number $\gamma'_{cc}(G)$ is the minimum cardinality of a complete cototal edge dominating set of G.*

3. Main Results

Theorem 3.1. *Let G be a connected graph. If D is a γ'_{cctd} - set of G, then $\langle E(G) - D \rangle$ is also a complete cototal edge dominating set.*

Proof. Let D be a γ'_{cctd} - set of G. Let us assume $\langle E(G) - D \rangle$ is not a $\gamma'_{cctd}-$ set of G. (i.e) no node belongs to e does not belongs to any edge in $\langle E(G) - D \rangle$. But then the set $\langle D - e \rangle$ should become a $\gamma'_{cctd}-$ set, which is a contradiction to the minimality of D. Hence $\langle E(G) - D \rangle$ is a $\gamma'_{cctd}-$ set of G. □

Theorem 3.2. *Every connected graph G contains a complete cototal edge dominating set and hence a complete cototal edge domination number.*

Proof. Let $G = (V, E)$ be a connected graph. Since every edge dominates to itself, the edge set $E(G)$ itself is $\gamma'_{cctd}(G)$. As G is nontrivial, every edge x is adjacent to some other edge y. Hence both x and y dominate x. Now we know that G has a $\gamma'_{cctd}-$ set. If we eliminate one edge at a time from $E(G - \{e\})$

then the remaining subset of E itself is $\gamma'_{cctd}(G)$ and also which is minimal. Then the minimal cardinality of a $\gamma'_{cctd}(G)$ is the complete cototal edge domination number $\gamma'_{cc}(G)$. □

Theorem 3.3. *For a Star graph $K_{1,n}$, $\gamma'_{cc}(K_{1,n}) = \begin{cases} 2 \text{ if } n = 2 \\ 3 \text{ if } n = 3 \\ 2 \text{ if } n \geq 4 \end{cases}$.*

Proof. The Star graph $K_{1,n}$ has $(n+1)$ nodes $u, v_1, v_2, ..., v_n$ and n edges uv_i, $1 \leq i \leq n$. Let u be the center node of $K_{1,n}$.

Case (i) $n = 2$.

The Star graph $K_{1,2}$ has three nodes u, v_1, v_2 and two edges uv_1, uv_2. Let us consider the total edge dominating set $\gamma'_{td}(K_{1,2}) = \{uv_1, uv_2\}$. Minimal cototal edge dominating set is obtained by $E(K_{1,2}) - \{uv_1, uv_2\}$. Therefore $\gamma'_{cctd}(K_{1,2}) = \{uv_1, uv_2\}$. Hence $\gamma'_{cc}(K_{1,2}) = 2$.

Case (ii) $n = 3$.

The Star graph $K_{1,3}$ has three nodes u, v_1, v_2, v_3 and three edges uv_1, uv_2 and uv_3. Take $\gamma'_{td}(K_{1,3}) = \{x\}$ where $x \in \{uv_1, uv_2\}$ or $\{uv_2, uv_3\}$ or $\{uv_1, uv_3\}$. Minimal cototal edge dominating set is obtained by $(E(K_{1,3}) - \{x\}) \bigcap \{y\}$, where y is an isolated edge. Therefore $\gamma'_{cctd}(K_{1,3}) = \{x\} \bigcup \{y\}$. Hence $\gamma'_{cc}(K_{1,3}) = 3$.

Case (iii) $n \geq 4$.

The Star graph $K_{1,n}$ has $(n+1)$ nodes $u, v_1, v_2, ..., v_n$ and n edges uv_i, $1 \leq i \leq n$. Take $\gamma'_{td}(K_{1,n}) = \{uv_i, uv_{i+1}\}$ where i can take any one of the value from 1 to $n-1$. Minimal cototal edge dominating set is obtained by $E(K_{1,n}) - \{uv_i, uv_{i+1}\}$. Therefore $\gamma'_{cctd}(K_{1,n}) = \{uv_i, uv_{i+1}\}$. Hence $\gamma'_{cc}(K_{1,n}) = 2$. □

Theorem 3.4. *For a Complete graph K_n, $\gamma'_{cc}(K_n) = \begin{cases} 3 \text{ if } n = 3 \\ n-2 \text{ if } n \geq 4 \end{cases}$.*

Proof. The Complete graph K_n has n nodes $v_1, v_2, ..., v_n$ and nC_2 edges.

Case (i) $n = 3$.

The Complete graph K_3 has three nodes v_1, v_2, v_3 and three edges v_1v_2, v_2v_3, v_3v_1. Take $\gamma'_{td}(K_3) = \{x\}$ where $x \in \{v_1v_2, v_2v_3\}$ or $\{v_2v_3, v_3v_1\}$ or $\{v_1v_2, v_3v_1\}$. Minimal cototal edge dominating set is obtained by $(E(K_3) - \{x\}) \bigcap \{y\}$ where y is an isolated edge. Therefore $\gamma'_{cctd}(K_3) = \{x\} \bigcup \{y\}$. Hence $\gamma'_{cc}(K_3) = 3$.

Case (ii) $n \geq 4$.

The Complete graph K_n has n nodes and nC_2 edges. Take $\gamma'_{td}(K_n) = \{v_1v_2, v_2v_3, ..., v_{n-2}v_{n-1}\}$. Then minimal cototal edge dominating set is obtained by $E(K_n) - \{v_1v_2, v_2v_3, ..., v_{n-2}v_{n-1}\}$.
Therefore $\gamma'_{cctd}(K_n) = \{v_1v_2, v_2v_3, ..., v_{n-2}v_{n-1}\}$. Hence $\gamma'_{cc}(K_n) = n - 2$. □

Theorem 3.5. *For a Comb graph $P_n \odot K_1 (n \geq 2)$, $\gamma'_{cc}(P_n \odot K_1) = 2n - 1$.*

Proof. The Comb graph $P_n \odot K_1$ has $2n$ nodes $v_1, v_2, ..., v_n, u_1, u_2, ..., u_n$ and $(2n - 1)$ edges $v_iv_{i+1}, 1 \leq i \leq n - 1$ and $v_iu_i, 1 \leq i \leq n$. Let $v_1, v_2, ..., v_n$ be the nodes of P_n and $u_1, u_2, ..., u_n$ be the pendant nodes. Take $\gamma'_{td}(P_n \odot K_1) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$. Minimal cototal edge dominating set is obtained by $(E(P_n \odot K_1) - \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}) \bigcap \{y\}$ where $y = \{v_iu_i\}$ for $1 \leq i \leq n$. Therefore $\gamma'_{cctd}P_n \odot K_1) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\} \bigcup \{y\}$. Hence $\gamma'_{cc}(P_n \odot K_1) = 2n - 1$. □

Theorem 3.6. *For a Friendship graph F_n, $\gamma'_{cc}(F_n) = \begin{cases} 3 \text{ if } n = 1 \\ n \text{ if } n \geq 2 \end{cases}$.*

Proof. The Friendship graph F_n has $(2n + 1)$ nodes $v_1, v_2, ..., v_{2n}, u$ and $3n$ edges.
Case (i) $n = 1$.
The Friendship graph F_1 has three nodes u, v_1, v_2 and three edges uv_1, uv_2, v_1v_2.
Let us consider the total edge dominating set $\gamma'_{td}(F_1) = \{uv_1, uv_2\}$ or $\{v_1v_2, uv_2\}$.
Minimal cototal edge dominating set is obtained by $(E(F_1) - \{uv_1, uv_2\}) \bigcap \{v_1v_2\}$ or $(E(F_1) - \{v_1v_2, uv_2\}) \bigcap \{uv_1\}$. Therefore $\gamma'_{cctd}(F_1) = \{uv_1, uv_2\} \bigcup \{v_1v_2\}$. Hence $\gamma'_{cc}(F_1) = 3$.
Case (ii) $n \geq 2$.
The Friendship graph F_n has $(2n + 1)$ nodes and $3n$ edges. Take $\gamma'_{td}(F_n) = X$ where X is the set of all edges taken from one edge each triangles which are incident with u and $|X| = n$. Minimal cototal edge dominating set is obtained by $E(F_n) - \{X\}$. Therefore $\gamma'_{cctd}(F_n) = \{X\}$. Hence $\gamma'_{cc}(F_n) = n$. □

Theorem 3.7. *For a Coconut tree graph $CT(m, n), m, n \geq 2$,*

$$\gamma'_{cc}(CT(m, n)) = \begin{cases} 3 \text{ if } m = n = 2 \\ 2 \text{ if } m = 2, n \geq 3 \\ m - 1 \text{ if } m, n \geq 3 \end{cases}.$$

Proof. The Coconut tree graph $CT(m, n)$ has $(m + n)$ nodes $u_1, u_2, ..., u_n, u$, $v_1, v_2, ..., v_{m-1}$ and $(m+n-1)$ edges uu_i, uv_1, v_jv_{j+1} and $1 \leq i \leq n, 1 \leq j \leq m-2$.

Here $u_1, u_2, ..., u_n$ be the pendant nodes of star $K_{1,n}$ and $u, v_1, v_2, ..., v_{m-1}$ be the nodes of path P_m with u as a common node.
Case (i) $m = n = 2$.
The Coconut tree graph $CT(2,2)$ has four nodes u_1, u_2, u, v_1 and three edges uu_1, uu_2, uv_1. Take $\gamma'_{td}(CT(2,2)) = \{x\}$ where $x \in \{uv_1, uu_1\}$ or $\{uu_2, uv_1\}$. Minimal cototal edge dominating set is obtained by $(E(CT(2,2)) - \{x\}) \bigcap \{y\}$ where y is an isolated edge. Therefore $\gamma'_{cctd}(CT(2,2)) = \{uv_1, uu_1\} \bigcup \{uu_2\}$. Hence $\gamma'_{cc}(CT(2,2)) = 3$.
Case (ii) $m = 2, n \geq 3$.
The Coconut tree graph $CT(2,n)$ has $(n+2)$ nodes $u_1, u_2, ..., u_n, u, v_1$ and $(n+1)$ edges $uu_i, uv_1, 1 \leq i \leq n$. Take $\gamma'_{td}(CT(2,n)) = \{uv_1, uu_i\}$ where i can take of any one of the value from 1 to n. Minimal cototal edge dominating set is obtained by $E(CT(2,n)) - \{uv_1, uu_i\}$. Therefore $\gamma'_{cctd}(CT(2,n)) = \{uv_1, uu_i\}$. Hence $\gamma'_{cc}(CT(2,n)) = 2$.
Case (iii) $m = 3, n \geq 2$.
The Coconut tree graph $CT(3,n)$ has $(n+3)$ nodes $u_1, u_2, ..., u_n, u, v_1, v_2$ and $(n+2)$ edges $uu_i, uv_1, v_1v_2, 1 \leq i \leq n$. Take $\gamma'_{td}(CT(3,n)) = \{uv_1, v_1v_2\}$. Minimal cototal edge dominating set is obtained by $E(CT(3,n)) - \{uv_1, v_1v_2\}$. Therefore $\gamma'_{cctd}(CT(3,n)) = \{uv_1, v_1v_2\}$. Hence $\gamma'_{cc}(CT(3,n)) = 2$.
Case (iv) $m, n \geq 3$.
The Coconut tree graph $CT(m,n)$ has $(m+n)$ nodes and $(m+n-1)$ edges.Take $\gamma'_{td}(CT(m,n)) = \{uv_1, v_iv_{i+1}\}$, where $1 \leq i \leq m-2$. Minimal cototal edge dominating set is obtained by $(E(CT(m,n)) - \{uv_1, v_iv_{i+1}\}) \bigcap \{y\}$, where y is an isolated edge. Therefore $\gamma'_{cctd}(CT(m,n)) = \{uv_1, v_iv_{i+1}\} \bigcup \{y\}$.
Hence $\gamma'_{cc}(CT(m,n)) = m - 1$. □

Theorem 3.8. *Let G be a connected graph of order n. Then $\gamma'_{cc}(G) \geq \left\lceil \frac{n}{\Delta(G)} \right\rceil$.*

Proof. Let S be a γ'_{cctd} - set of G. Then, we know that for every $e \in G$ is adjacent to some e of S. (i.e) $N(S) = E(G)$. As every $e \in S$ can have at most Δ neighbours, then $\Delta\gamma'_{cc}(G) > |E| = n$. Hence $\gamma'_{cc}(G) \geq \left\lceil \frac{n}{\Delta(G)} \right\rceil$. □

Result 1. *The above bound is sharp for $K_{1,n}(n \neq 3)$ since $\gamma'_{cc}(K_{1,n}) = 2$.*

Theorem 3.9. *For a graph G of order $n \geq 3$ with $diam(G) \geq 2$, $\gamma'_{cc}(G) \geq \delta(G)+1$ iff G is not a Complete graph or a Star graph ($n \geq 4$).*

Proof. Let $t \in E(G)$ and $deg(t) = \delta(G)$. Since $diam(G) \geq 1$, then $N(t)$ is a total edge dominating set for G. Now $S = N(t) \bigcup \{t\}$ is a complete cototal edge dominating set for G and $|S| = \delta(G) + 1$. Hence, $\gamma'_{cc}(G) \geq \delta(G) + 1$.

Conversely, Suppose $G = K_n (n \geq 4)$ be a Complete graph with $diam(G) \geq 2$. By Theorem 3.4, $\gamma'_{cc}(G) = n - 2$. We know that $\delta(G) \geq 2$. Hence $\gamma'_{cc}(G) \leq \delta(G) + 1$.

Assume $G = K_{1,n} (n \geq 4)$ be a Star graph with $diam(G) \geq 2$. By Theorem 3.3, $\gamma'_{cc}(G) = 2$. We know that $\delta(G) \geq 2$. Hence $\gamma'_{cc}(G) \leq \delta(G) + 1$. □

Result 2. *The above bound is sharp for* $CT(3, n)(n \geq 2)$, $K_{1,2}$, $K_{1,3}$, F_3 *and* K_3 *since* $\gamma'_{cc}(CT(3, n)) = 2$, $\gamma'_{cc}(K_{1,2}) = 2$, $\gamma'_{cc}(K_{1,3}) = 3$, $\gamma'_{cc}(K_3) = 3$ *and* $\gamma'_{cc}(F_3) = 3$.

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