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DIFFERENCE SCHEME FOR THE NUMERICAL SOLUTION OF DELAY DIFFERENTIAL EQUATIONS WITH LAYER STRUCTURE

D. KUMARA SWAMY, M. ADILAXMI, AND G. B. S. L. SOUJANYA[1]

ABSTRACT. A difference scheme is presented for the numerical solution of delay differential equations having layer structure. Firstly, the given delay differential equation is replaced by singularly perturbed two-point boundary value problem using Taylor's expansion on delay/deviating term. Next, Liouville Green Transformation is used to convert into regularly perturbed two-point boundary value problem. This problem is solved efficiently by the finite difference scheme of order four. The scheme presented here is implemented on four model problems for different values of perturbation and delay parameters. The numerical solutions are compared with exact solutions and other results available in literature. To understand the impact of the parameters, the solution is also shown in graphs.

1. INTRODUCTION

In the modelling of complex physical systems, differential-difference equations play a very important role. In particular, to get practical feedback models, it is always important to have delaying effects, such as reaction time. These problems occur in the modelling of many practical phenomena such as, in population dynamics Kuang [9], in models for physiological processes [14], predator-prey models [15] and in evaluating of the estimated time for the generation of

[1]*corresponding author*

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an action potential in nerve cells by random synaptic inputs in dendrites [21]. For further analysis of mathematical details of these models, researchers can refer to Derstine et al. [2], El'sgol'ts and Norkin [4]. Kokotovic et al. [7]. In the last few years, the study of differential-difference equations with layer behaviour having small shifts has evolved rapidly. Most of the researchers solved these equations by discretization using computational techniques. The analysis and numerical approach of these problems are well documented in [1, 3, 7, 8, 17, 18]. Authors in [6], using a parameter-uniform differential scheme with the use of Taylor approximation, solved a mathematical model resulting from the neuronal variability model. In [10]. authors studied a second-order class differential-difference equations that exhibit turning point behaviour and in [11] concentrated on solutions that display layer behaviour at one or both ends of the boundary and evaluated the layer behaviour for different values of the shift parameter. The same authors in [12] expanded their analysis to problems with fast oscillation solutions and demonstrated that oscillatory solutions are more prone to small delays than layer solutions using a simplified version of the standard WKB process. In [13], a mixed difference method is suggested to solve the equations with mixed shifts via domain decomposition as an inner and outer region. Researchers in [16] proposed initial value method for solving a class of differential-difference equations with mixed shifts, first changing the given problem into singular perturbation problem using Taylors series and splitting it into two explicit initial value problems that are independent of perturbation parameter and solved them numerically, Phaneendra et. al. [18] derived a fitting factor finite difference method of fourth order to solve delay problems with mixed shifts. In [22], a set of finite difference methods proposed for convention-diffusion equations using triangular function theorem for one-dimensional problems and the same is generalized for 2D problems with the help AID technique. Ravi Kanth and Murali [19] presented a fitted spline scheme for the solution of convection delay problems with a layer at left or right end of the interval. Hussein Sahihi et. al. [5] used Reproducing Kernel Hilbert Space Method based on collocation scheme is used without Gram-Schmidt orthogonalization process for solving differential-difference equation with boundary layer behavior and also oscillatory behavior with small delay. Kumara Swamy et. al. [10] employed numerical integration with linear interpolation to get the solution of differential equations with layer or oscillatory structure. Reddy et al.

[20] implemented Trapezoidal rule for the solution of layered behaviour differential equations.

2. Description of the Problem

We consider the second order linear differential equation with delay argument

$$\varepsilon w''(\vartheta) + a(\vartheta)w'(\vartheta - \delta) + b(\vartheta)w(\vartheta) = 0, \ \ 0 \leq \vartheta \leq 1\,, \tag{2.1}$$

with the boundary conditions

$$w(\vartheta) = \alpha \quad , -\delta \leq \vartheta \leq 0 \text{ and } \quad w(1) = \beta\,,$$

where $0 < \varepsilon << 1$ is the perturbation parameter, $0 < \delta < 1$ and $\delta = o(\varepsilon)$ is the delay argument, $a(\vartheta)$, $b(\vartheta)$ sufficiently differentiable in (0,1) and α, β are constants. When $a(\vartheta) \geq M > 0$ in [0, 1], boundary layer will be in the neighborhood of $\vartheta = 0$ and when $a(\vartheta) \leq M < 0$ in [0, 1], boundary layer will be in the neighborhood of $\vartheta = 1$. Taylor series expansion gives us

$$w^{'}(\vartheta - \delta) \approx w^{'}(\vartheta) - \delta w^{''}(\vartheta)\,. \tag{2.2}$$

Using equation (2.2) into equation (2.1), we obtain singularly perturbed equation as:

$$-\varepsilon w^{''}(\vartheta)+f(\vartheta)w^{'}(\vartheta)+g(\vartheta)w(\vartheta) = 0 \tag{2.3}$$

where $f(\vartheta) = \frac{a(\vartheta)}{\tau a(\vartheta)-1}$, $g(\vartheta) = \frac{b(\vartheta)}{\tau a(\vartheta)-1}$, $\tau = \frac{\delta}{\varepsilon}$. Since $0 < \delta < 1$ the transition from equation (2.1) to equation (2.3) is permitted. Justification for this is available in Elsgolt's and Norkin [4].

3. Numerical Scheme

Consider the equation (2.3)

$$-\varepsilon w^{''}(\vartheta) + f(\vartheta)w^{'}(\vartheta) + g(\vartheta)w(\vartheta) = 0\,, \ \vartheta \in [0, 1]\,. \tag{3.1}$$

The Liouville –Green transformation is given by:

$$z = \varphi(\vartheta) = \frac{1}{\varepsilon}\int f(\vartheta)d\vartheta = \varphi\,(\vartheta) = \frac{1}{\varepsilon}\int \ ,\mathrm{f}\,(\vartheta)\,\mathrm{d}\vartheta\,, \tag{3.2}$$

$$\phi(\vartheta) = \varphi'(\vartheta) = \frac{1}{\varepsilon}f(\vartheta)\,.$$

$$l(z) = \phi(\vartheta) w(\vartheta)\,. \tag{3.3}$$

According equation (3.3), we have

$$\frac{dw}{d\vartheta} = \frac{1}{\phi(\vartheta)}\frac{dl}{dz}z^{'}(\vartheta) - \frac{\phi^{'}(\vartheta)}{\phi^2(\vartheta)}l(z) = \frac{\phi^{'}(\vartheta)}{\phi(\vartheta)}\frac{dl}{dz} - \frac{\phi^{'}(\vartheta)}{\phi^2(\vartheta)}l(z), \tag{3.4}$$

$$\frac{d^2w}{d\vartheta^2} = \frac{1}{\phi(\vartheta)}\left(\left(\varphi^2(\vartheta)\frac{d^2l}{dz^2} + \left(\phi'' - \frac{2\varphi^{'}(\vartheta)\phi^{'}(\vartheta)}{\phi(\vartheta)}\right)\frac{dl}{dz}\right) - \left(\frac{\phi^{''}(\vartheta)}{\phi(\vartheta)} - \frac{2\varphi^{'2}(\vartheta)}{\phi^2(\vartheta)}l\right)\right)\,. \tag{3.5}$$

From equation (3.1), equation (3.4)and equation (3.5), we obtain

$$-\frac{\varepsilon\varphi^{'2}}{\phi}\frac{d^2l}{dz^2} + \left(\frac{2\varepsilon\varphi^{'}\phi^{'}}{\phi^2} - \frac{\varepsilon\varphi^{''}(\vartheta)}{\phi(\vartheta)} + f(s)\frac{\varphi^{'}(\vartheta)}{\phi(\vartheta)}\right)\frac{dl}{dz}$$
$$+\left(\frac{\varepsilon\phi^{''}(\vartheta)}{\phi^2(\vartheta)} - \frac{2\varepsilon\varphi^{'2}(\vartheta)}{\phi^3(\vartheta)} - f(\vartheta)\frac{\phi^{'}(\vartheta)}{\phi^2} + \frac{g(s)}{\phi}\right)l(z) = 0,$$

i.e.,

$$\frac{d^2l}{dz^2} + \frac{1}{\varphi^{'2}}\left(\varphi^{''}(\vartheta) - \frac{2\varphi^{'}\phi^{'}}{\phi} - f(\vartheta)\frac{\varphi^{'}(\vartheta)}{\varepsilon}\right)\frac{dl}{dz}$$
$$-\frac{1}{\varphi^{'2}}\left(\frac{\phi^{''}(\vartheta)}{\phi(\vartheta)} - \frac{2\phi^{'2}}{\phi^2} - f(\vartheta)\frac{\phi^{'}(\vartheta)}{\varepsilon\phi(\vartheta)} + \frac{g(\vartheta)}{\varepsilon}\right)l(z) = 0\,.$$

From equation (3.2), we have

$$\frac{d^2l}{dz^2} - \left(\varepsilon\frac{f^{'}(\vartheta)}{f^2(\vartheta)} + 1\right)\frac{dl}{dz} - \frac{1}{f^2(\vartheta)}\left(\varepsilon^2\frac{f^{''}(\vartheta)}{f(\vartheta)} - 2\varepsilon^2\frac{f^{'2}(\vartheta)}{f^2(\vartheta)} - \varepsilon f^{'}(\vartheta) + \varepsilon g(\vartheta)\right)l(z) = 0,$$

$$\frac{d^2l}{dz^2} - \left(\varepsilon\frac{f^{'}(\vartheta)}{f^2(\vartheta)} + 1\right)\frac{dl}{dz} - \varepsilon\left(-\frac{f^{'}(\vartheta)}{f^2(\vartheta)} + \frac{g(\vartheta)}{f^2(\vartheta)}\right)l(z) = \frac{\varepsilon^2}{f^2(\vartheta)}\left(\frac{f^{''}(\vartheta)}{f(\vartheta)} - 2\frac{f^{'2}(\vartheta)}{f^2(\vartheta)}\right)l(z)\,,$$

$$\frac{d^2l}{dz^2} - p(\vartheta)\frac{dl}{dz} - \varepsilon q(\vartheta)l(z) = \frac{\varepsilon^2}{f^2(\vartheta)}\left(\frac{f^{?}(\vartheta)}{f(\vartheta)} - 2\frac{f^{'2}(\vartheta)}{f^2(\vartheta)}\right)l(z)\,, \tag{3.6}$$

where $p(\vartheta) = \varepsilon\frac{f^{'}(\vartheta)}{f^2(\vartheta)} + 1,\ q(\vartheta) = \frac{1}{f^2(\vartheta)}\left(-f^{'}(\vartheta) + g(\vartheta)\right)$. Since ε is a small parameter $(0 < \varepsilon << 1)$, right hand side of equation (3.6) is sufficiently small on $[0, 1]$.

$$\frac{d^2l}{dz^2} - p(\vartheta)\frac{dl}{dz} - \varepsilon q(\vartheta)l(z) = 0\,. \tag{3.7}$$

The boundary conditions for the problem equation (3.7) are given by equation (3.3). We have the following difference approximations for l_i', l_i'':

(3.8) $$l_i^{''} \cong \frac{l_{i-1} - 2l_i + l_{i+1}}{h^2} - \frac{h^2}{12} l^4(\xi) + R_2\,,$$

(3.9) $$l_i^{'} \cong \frac{l_{i+1} - l_{i-1}}{2h} - \frac{h^2}{6} l^{(2)}(\eta) + R_1\,,$$

where $\xi, \eta \in [\vartheta_{i-1}, \vartheta_{i+1}]$ $R_1 = \frac{-2h^4 l^{(3.1)}(\xi_1)}{120} R_2 = \frac{-2h^4 l^{(3.2)}(\xi_2)}{240}$

$$\frac{l_{i-1} - 2l_i + l_{i+1}}{h^2} - \frac{h^2}{12} l^{(2.3)}(\xi) + R_2 - p_i \left(\frac{l_{i+1} - l_{i-1}}{2h} \right) + \frac{p_i h^2}{6} l_i^{'''} + p_i R_1 - \varepsilon q_i l_i = 0\,.$$

Now differentiating both sides of equation (3.7), we have

$$l^{'''} = pl^{''} + (p^{'} + \varepsilon q) l^{'} + \varepsilon q^{'} l\,.$$

Using the above equations, we have:

(3.10)
$$\frac{l_{i-1} - 2l_i + l_{i+1}}{h^2} - p_i \left(\frac{l_{i+1} - l_{i-1}}{2h} \right) + \frac{p_i h^2}{6} \left(p_i l_i^{''} + (p_i^{'} + \varepsilon q_i) l_i^{'} + \varepsilon q_i^{'} l_i \right) - \varepsilon q_i l_i + R = 0\,.$$

where $R = p_i R_1 - \frac{h^2}{12} l_i^{iv} + R_2$. Now we approximate the converted error term in equation (3.10) by using difference formulas for $l_i^{'}, l_i^{''}$ from equation (3.8)and equation (3.9). Then we get

$$\left(\frac{1}{h^2} + \frac{p_i}{2h} + \frac{p_i^2}{6} - \frac{p_i h}{12}(p_i^{'} + \varepsilon q_i) \right) l_{i-1} - \left(\frac{2}{h^2} + \frac{p_i^2}{3} - \frac{\varepsilon p_i h^2}{6} q_i^{'} + \varepsilon q_i \right) l_i$$
$$+ \left(\frac{1}{h^2} - \frac{p_i}{2h} + \frac{p_i^2}{6} + \frac{p_i h}{12}(p_i^{'} + \varepsilon q_i) \right) l_{i+1} + \tau_i(l) = 0\,,$$

where $\tau_i(l) = \frac{p_i^2 h^2}{6} R_2 - \frac{p_i^2 h^4}{6} l_i^{iv} - \frac{p_i h^4}{36}(p_i^{'} + \varepsilon q_i) l_i^{'''} + \frac{p_i h^2}{6}(p_i^{'} + \varepsilon q_i) R_1 + R$. Simplifying the above equation, we get the three term relation given by:

(3.11) $$\mathrm{A}_i l_{i-1} - B_i l_i + C_i l_{i+1} = D_i, i = 1, 2, ..., N-1$$

where $A_i = \frac{1}{h^2} + \frac{p_i}{2h} + \frac{p_i^2}{6} - \frac{p_i h}{12}(p_i^{'} + \varepsilon q_i), \quad B_i = \frac{2}{h^2} + \frac{p_i^2}{3} - \frac{\varepsilon p_i h^2}{6} q_i^{'} + \varepsilon q_i$

$C_i = \frac{1}{h^2} - \frac{p_i}{2h} + \frac{p_i^2}{6} + \frac{p_i h}{12}(p_i^{'} + \varepsilon q_i), \quad D_i = 0.$

equation (3.11) is solved for the solution using tridiagonal solver algorithm.

4. NUMERICAL EXPERIMENTS

Approach presented here is implemented on four model problems for different values of ε and δ and our solutions are compared with exact solutions available in literature. To understand the impact of the parameters the solution is also shown in graph. Wherever the exact solution is not available, Error is calculated using the double mesh principle given by $E^N = \max_{0\le i\le N} \left| w_i^N - w_{2i}^{2N} \right|$.

Problem 4-1. Consider a delay differential equation with left-end layer

$$\varepsilon w''(\vartheta) + w'(\vartheta - \delta) - w(\vartheta) = 0\,; \quad \vartheta \in [0,1]\,.$$

The exact solution is:

$$w(\vartheta) = \frac{((1-e^{m_2})e^{m_1\vartheta} + (e^{m_1} - 1)e^{m_2\vartheta})}{(e^{m_1} - e^{m_2})}\,,$$

where $m_1 = \dfrac{-1 - \sqrt{1 + 4(\varepsilon - \delta)}}{2(\varepsilon - \delta)}$ and $m_2 = \dfrac{-1 + \sqrt{1 + 4(\varepsilon - \delta)}}{2(\varepsilon - \delta)}$.

Maximum errors are compared in Table 1 and Table 2 using double mesh principle.

Problem 4-2. $\varepsilon w''(\vartheta) + e^{-0.5\vartheta} w'(\vartheta - \delta) - w(\vartheta) = 0$ with $w(0) = 1,\ w(1) = 1.$

Maximum errors are compared in Table 3 and Table 4 using double mesh principle.

Problem 4-3. Consider a delay differential equation with right-end layer

$$\varepsilon w''(\vartheta) - w'(\vartheta - \delta) - w(\vartheta) = 0 \text{ with } w(0) = 1,\ w(1) = -1\,.$$

The exact solution is

$$w(\vartheta) = \frac{((1 + e^{m_2})e^{m_1\vartheta} - (e^{m_1} + 1)e^{m_2\vartheta})}{(e^{m_2} - e^{m_1})}$$

where $m_1 = \dfrac{-1 - \sqrt{1 + 4(\varepsilon - \delta)}}{2(\varepsilon - \delta)}$ and $m_2 = \dfrac{-1 + \sqrt{1 + 4(\varepsilon - \delta)}}{2(\varepsilon - \delta)}$.

Maximum errors are compared in Table 5 and Table 6 using double mesh principle.

Problem 4-4. $\varepsilon w''(\vartheta) - e^{\vartheta} w'(\vartheta - \delta) - \vartheta w(\vartheta) = 0,$ with $w(0) = 1, w(1) = 1.$

Maximum errors are compared in Table 7 and Table 8 using double mesh principle.

5. Conclusion

This paper dealt with the delay differential equation having boundary layers. The delay argument is dealt by Taylor's series and a singularly perturbed problem is derived. Liouville Green Transformation and higher order finite difference method is described for solving the resulting problem. Approach presented here is implemented on four model problems for different values of ε and δ and our solutions are compared with exact solutions and with the results available in [20]. To understand the impact of the parameters the solution is also shown in graphs.

Table 1. The maximum absolute errors in solution of Problem 1 with $\epsilon = 0.1$ for different values of δ and grid size N.

$N \rightarrow$	10^2	10^3	10^4	10^5
$\delta \downarrow$		Proposed method		
0.01	$1.3798e-04$	$1.3907e-05$	$1.3887e-06$	$1.5767e-06$
0.03	$1.0765e-04$	$1.0849e-05$	$1.0600e-06$	$3.1523e-06$
0.06	$6.1798e-05$	$6.2273e-06$	$6.3164e-07$	$3.1727e-06$
0.08	$3.0995e-05$	$3.1233e-06$	$3.3537e-07$	$1.5918e-06$
		Results in [20]		
0.01	0.01172504	0.00122562	$1.2310e-004$	$1.2280e-05$
0.03	0.01505997	0.00158944	$1.5984e-004$	$1.5998e-05$
0.06	0.02575368	0.00281263	$2.8397e-004$	$2.8449e-05$
0.08	0.04781066	0.00562948	$5.7357e-004$	$5.7357e-004$

Table 2. The maximum absolute errors in solution of Problem 1 with $\epsilon = 0.01$ for different values of δ and grid size N.

$N \rightarrow$	10^2	10^3	10^4	10^5
$\delta \downarrow$		Proposed method		
0.01	$1.5388e-04$	$1.5487e-05$	$1.5676e-06$	$1.5953e-06$
0.03	$1.1972e-04$	$1.2049e-05$	$1.2321e-06$	$1.5919e-06$
0.06	$6.8442e-05$	$6.8883e-06$	$7.2214e-07$	$1.5895e-06$
0.08	$3.4231e-05$	$3.4452e-06$	$3.6130e-07$	$1.7706e-09$
		Results in [20]		
0.01	0.09073569	0.01228700	0.00127926	$1.28459e-04$
0.03	0.10803507	0.01562216	0.00164450	$1.65330e-04$
0.06	0.12777968	0.02630926	0.00287019	$2.89704e-04$
0.08	0.10040449	0.04833890	0.00568876	$5.79477e-04$

Table 3. The maximum absolute errors in solution of Problem 1 with $\epsilon = 0.1$ for different values of δ and grid size N.

$N \rightarrow$	10^2	10^3	10^4
$\delta \downarrow$	Proposed method		
0.01	$5.0306e-05$	$4.9837e-06$	$4.9792e-07$
0.03	$3.8174e-05$	$3.7549e-06$	$3.7476e-07$
0.06	$1.9104e-05$	$1.7998e-06$	$1.7879e-07$
0.08	$4.3277e-06$	$5.0732e-07$	$5.1673e-08$
	Results in [20]		
0.01	0.00632996	0.000674268	$6.7871e-05$
0.03	0.00815917	0.000882563	$8.8986e-05$
0.06	0.01384760	0.001579726	$1.6020e-04$
0.08	0.02477158	0.003173235	$3.2602e-04$

Table 4. The maximum absolute errors in solution of Problem 1 with $\epsilon = 0.01$ for different values of δ and grid size N.

$N \rightarrow$	10^2	10^3	10^4	10^5
$\delta \downarrow$	Proposed method			
0.01	$5.2697e-05$	$5.2414e-06$	$2.3268e-09$	$5.2312e-07$
0.03	$3.9647e-05$	$3.9249e-06$	$3.4353e-09$	$3.9256e-07$
0.06	$1.9325e-05$	$1.8582e-06$	$7.7315e-09$	$1.8664e-07$
0.08	$3.4.5115e-06$	$5.3175e-07$	$1.9107e-08$	$5.4691e-08$
	Results in [20]			
0.01	0.09092877	0.01562620	0.00127940	0.00012847
0.03	0.10836214	0.01562216	0.00164463	0.00016534
0.06	0.12845428	0.02631484	0.00287030	0.00028971
0.08	0.10149957	0.04834773	0.00568891	0.00057948

Table 5. The maximum absolute errors in solution of Problem 1 with $\epsilon = 0.1$ for different values of δ and grid size N.

$N \rightarrow$	10^2	10^3	10^4	10^5
$\delta \downarrow$	Proposed method			
0.01	$4.9650e-05$	$4.9729e-06$	$4.9586e-07$	$2.7422e-10$
0.03	$5.8439e-05$	$5.8534e-06$	$5.8693e-07$	$4.5744e-07$
0.06	$7.1489e-05$	$7.1607e-06$	$7.2219e-07$	$4.5575e-07$
0.08	$8.0100e-05$	$8.0235e-06$	$8.0780e-07$	$9.0628e-07$
	Results in [20]			
0.01	0.02281050	0.00236357	$2.3722e-004$	$2.3756e-05$
0.03	0.01954096	0.00201453	$2.0208e-004$	$2.0239e-05$
0.06	0.01609366	0.00165114	$1.6554e-004$	$1.6580e-05$
0.08	0.01439633	0.00147352	$1.4770e-004$	$1.4818e-05$

Table 6. The maximum absolute errors in solution of Problem 1 with $\epsilon = 0.1$ for different values of δ and grid size N.

$N \rightarrow$	10^2	10^3	10^4	10^5
$\delta \downarrow$		Proposed method		
0.01	$5.5816e-05$	$5.5822e-06$	$5.5361e-07$	$7.6147e-10$
0.03	$6.5938e-05$	$6.5944e-06$	$6.5969e-07$	$4.5744e-07$
0.06	$8.1106e-05$	$8.1113e-06$	$8.0942e-07$	$9.3737e-07$
0.08	$9.1208e-05$	$9.1214e-06$	$9.1528e-07$	$4.6885e-07$
		Results in [20]		
0.01	0.16595983	0.02210942	0.00228566	$2.29353e-004$
0.03	0.10803507	0.01562216	0.00164450	$1.94192e-004$
0.06	0.12777968	0.02630926	0.00287019	$2.89704e-004$
0.08	0.10040449	0.04833890	0.00568876	$5.79477e-004$

Table 7. The maximum absolute errors in solution of Problem 1 with $\epsilon = 0.1$ for different values of δ and grid size N.

$N \rightarrow$	10^2	10^3	10^4
$\delta \downarrow$	Proposed method		
0.01	$6.8368e-07$	$5.9728e-08$	$7.9893e-09$
0.03	$4.7333e-07$	$4.0201e-08$	$5.9759e-09$
0.06	$2.0459e-07$	$1.5284e-08$	$4.8387e-11$
0.08	$4.4650e-08$	$2.2655e-09$	$1.9286e-11$
	Results in [20]		
0.01	0.00575975	0.00050842	$5.0247e-005$
0.03	0.003932768	0.00036132	$3.5838e-005$
0.06	0.002702569	0.00025507	$2.5364e-005$
0.08	0.00224689	0.00021413	$2.1313e-005$

Table 8. The maximum absolute errors in solution of Problem 1 with $\epsilon = 0.1$ for different values of δ and grid size N.

$N \rightarrow$	10^2	10^3	10^4	10^5
$\delta \downarrow$		Proposed method		
0.0007	$7.8866e-08$	$6.9851e-09$	$8.1822e-11$	$6.3081e-12$
0.0015	$6.8474e-08$	$6.0053e-09$	$2.8251e-11$	$4.8875e-12$
0.0025	$5.6994e-08$	$8.1113e-06$	$8.0942e-07$	$7.0181e-08$
		Results in [20]		
0.0007	0.16595983	0.02210942	0.00301195	0.00030240
0.0015	0.12311973	0.01462776	0.00149178	0.00014948
0.0025	0.08096456	0.00911534	0.00092344	0.00009247

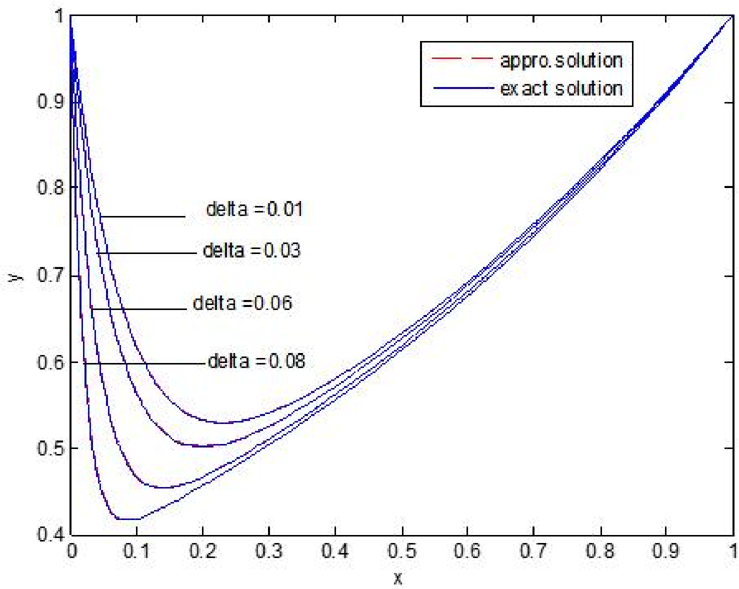

FIGURE 1. Solution of the Problem 4.1 for $\epsilon = 0.1$ for different δ of $o(\epsilon)$

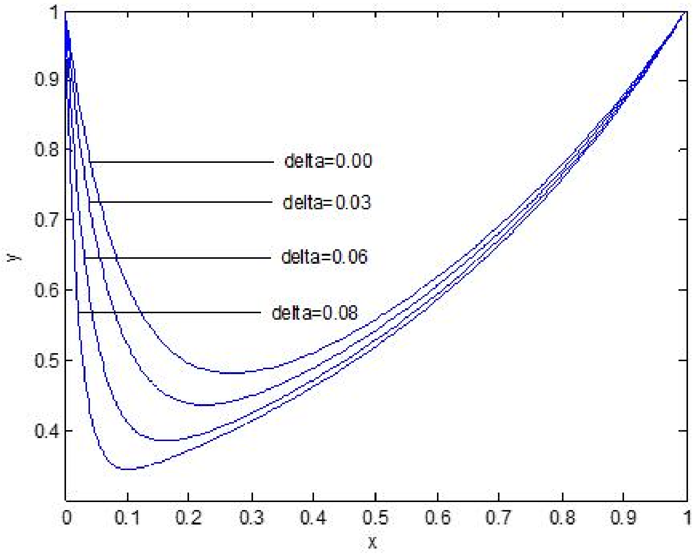

FIGURE 2. Solution of the Problem 4.1 for $\epsilon = 0.01$ for different δ of $o(\epsilon)$

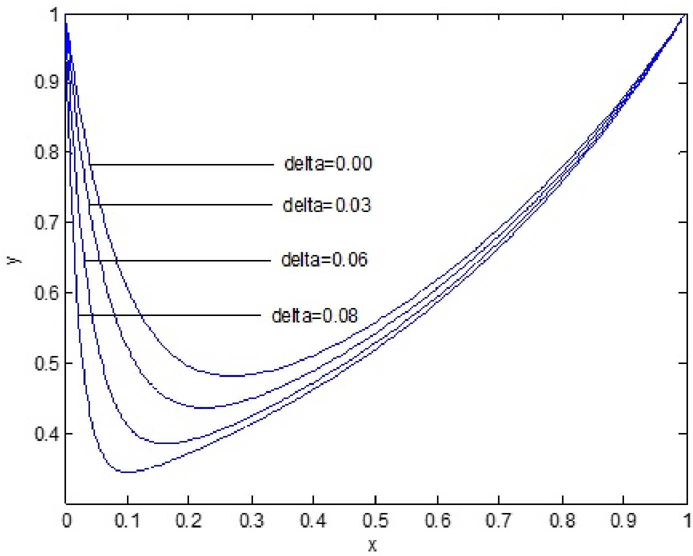

FIGURE 3. Solution of the Problem 4.2 for $\epsilon = 0.1$ for different δ of $o(\epsilon)$

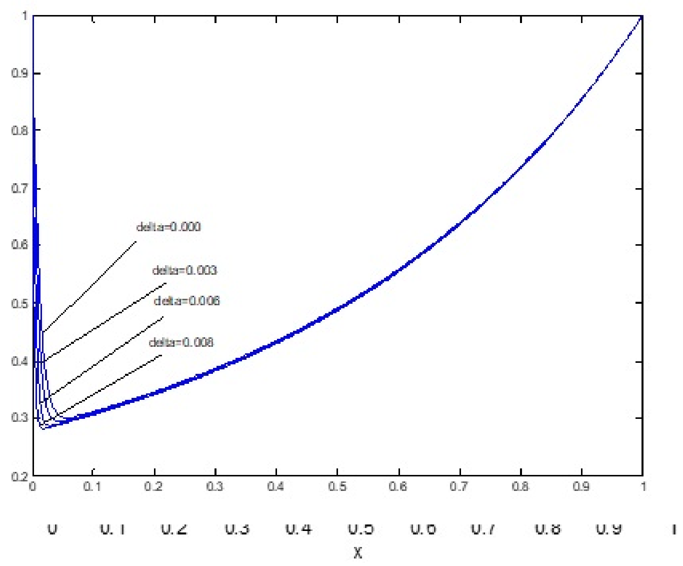

FIGURE 4. Solution of the Problem 4.2 for $\epsilon = 0.01$ for different δ of $o(\epsilon)$

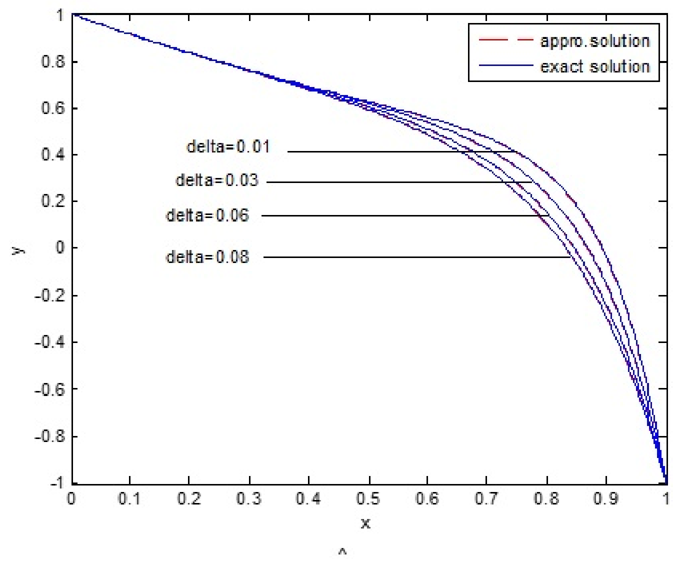

FIGURE 5. Solution of the Problem 4.3 for $\epsilon = 0.1$ for different δ of $o(\epsilon)$

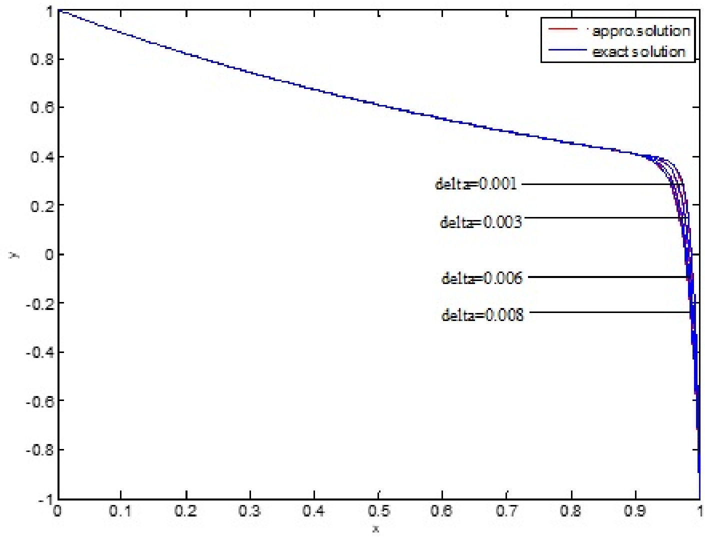

FIGURE 6. Solution of the Problem 4.3 for $\epsilon = 0.01$ for different δ of $o(\epsilon)$

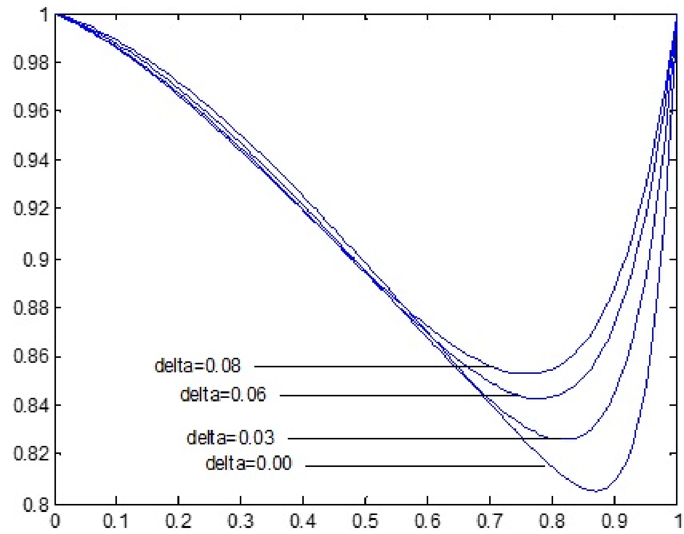

FIGURE 7. Solution of the Problem 4.4 for $\epsilon = 0.1$ for different δ of $o(\epsilon)$

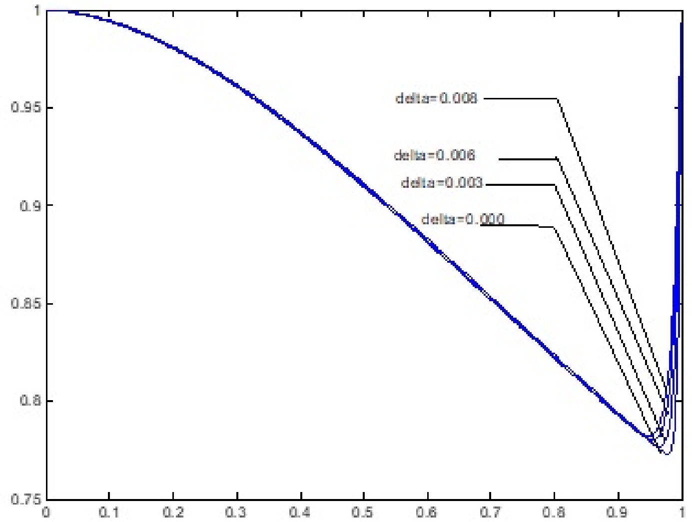

FIGURE 8. Solution of the Problem 4.4 for $\epsilon = 0.01$ for different δ of $o(\epsilon)$

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DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF INFORMATION TECHNOLOGY, SONEPAT
E-mail address: diddi.k@gmail.com

DEPARTMENT OF MATHEMATICS
KL UNIVERSITY, HYDERABAD
E-mail address: madireddyadilaxmi@gmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY ARTS AND SCIENCE COLLEGE
KAKATIYA UNIVERSITY, WARANGAL
E-mail address: gbslsoujanya@gmail.com