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SOME RESULTS ON ANTI-DUPLICATION OF A VERTEX IN GRAPHS

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ABSTRACT. For a finite undirected simple graph $G(V, E)$, duplication of a vertex $v \in V(G)$ forms a new graph G' by introducing a new vertex v' such that $N_{G'}(v') = N_G(v)$. We define anti-duplication of a vertex v in G by introducing a new vertex v' which produces a new graph G' such that $N_{G'}(v') = N_G(v)$. In this paper, we study its properties and characterize a graph to be connected, disconnected, regular and eulerian after anti-duplication of a vertex v in G.

1. INTRODUCTION

By a graph G we mean a finite undirected simple graph. The subgraph of G obtained by removing the vertex v and all the edges incident with v is called the subgraph obtained by the removal of the vertex v and is denoted by $G^{\smile} - v$. A closed walk that contains all edges of the graph is called an Euler line. A graph that contains an Euler line is called an Euler graph or Eulerian graph. A connected graph is Eulerian if and only if all the vertices are of even degree [4]. A graph G is regular of degree r if all vertices of G have the same degree r. A subgraph B of G which contains v is called a branch at v in G if $B - -v$ is connected and maximal [1]. The set of vertices adjacent to v in G is denoted by $N_G(v)$, the neighbours of v in G.

Switching in graphs was introduced by Lint and Seidel [5]. For a finite undirected graph $G(V, E)$ and a subset $S \subset V$, the switching of G by S is defined as

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the graph $G^S(V, E')$ which is obtained from G by removing all edges between S and its complement $V - \breve{S}$ and adding as edges all non edges between S and $V - \breve{S}$. For $S = \{v\}$, we write G^v instead of $G^{\{v\}}$ and the corresponding switching is called as vertex switching [6]. Duplication of a vertex v of graph G produces a new graph G' by adding a new vertex v' such that $N_{G'}(v') = N_G(v)$. In other words a vertex v' is said to be duplication of v if all the vertices which are adjacent to v in G are also adjacent to v' in G' [2]. In [3], E. Sampathkumar introduced duplicate graphs. For a given graph G with vertex set V, the duplicate graph DG is defined on the vertex set $V \cup V'$, where $V' = \{v'; v \in V\}$ and the edges are given as follows. ab is an edge in G if and only if both ab' and $a'b$ are edge in DG. The graph DG is called the duplicate of G. He also studied its properties. In this paper, we introduce the concept of anti-duplication of a vertex v in G and study its properties. We also characterize a graph to be connected, disconnected, regular and eulerian after anti-duplicating the vertex v in G.

2. Anti-duplication of a vertex in a Graph

Definition 2.1. *Anti-duplication of a vertex v in G produces a new graph G' by adding a new vertex v' such that $N_{G'}(v') = [N_G[v]]^c$.*

The graph obtained from G after anti-duplication of the vertex v is denoted by $AD(vG)$.

Example 1. *Consider the graph G given in figure 2.1. The anti-duplication of the vertices v_1,v_2, v_3 and v_4 are given in figures 2.2, 2.3, 2.4 and 2.5, respectively.*

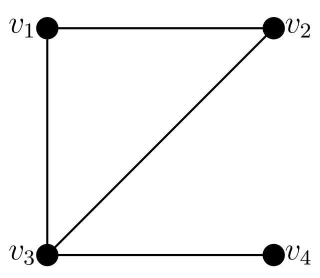

Fig 2.1. G

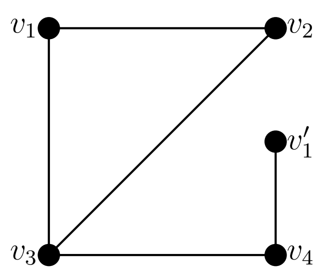

Fig 2.2. $AD(v_1G)$

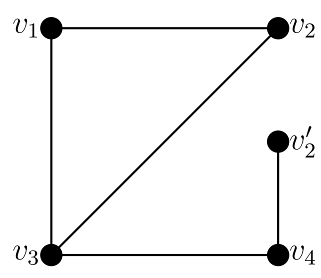

Fig 2.3. $AD(v_2G)$

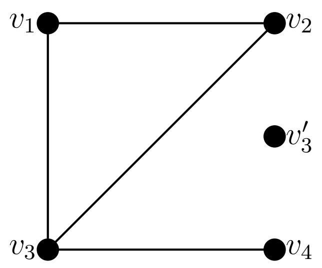

Fig 2.4. $AD(v_3G)$

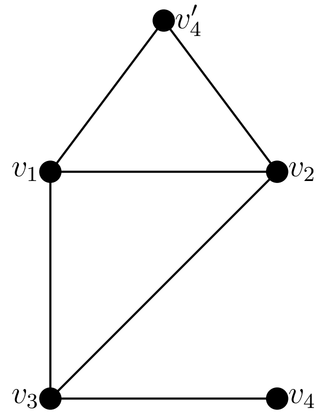

Fig 2.5. $AD(v_4G)$

Let G be a graph and $AD(vG)$ be the graph obtained by anti-duplication of the vertex v in G. In this paper we use the following notations for our convenience.

- $d(v) = d_G(v)$, the degree of the vertex v in the graph G ;
- $d'(v) = d_{AD(vG)}(v)$, the degree of the vertex v in the graph $AD(vG)$;
- $N(v) = N_G(v)$, the neighborhood of v in the graph G ;
- $N'(v) = N_{AD(vG)}(v)$, the neighborhood of v in the graph $AD(vG)$;
- $N[v] = N(v) \cup \{v\}$;
- $[N[v]]^c$, the vertices which are not the neighbours of v in G .

Theorem 2.1. *Let G be a graph and v be any vertex in G. Let v' be the anti-duplication vertex of v in $AD(vG)$. Then $AD(vG) - \breve{}\{v\} \cong G^v$, where G^v is the graph obtained by switching the vertex v in G.*

Proof. The graph $AD(vG)$ is obtained by adding a new vertex v' such that $N_{G'}(v') = [N_G[v]]^c$ and hence $N_G(v) = N_{G'}(v)$. Now $AD(vG)\breve{} - \{v\}$ is isomorphic to the graph G^v obtained by switching the vertex v in G as G^v is the graph obtained from G by removing all edges between v and its complement $V\breve{} - \{v\}$ and adding as edges all non edges between $V - \breve{}\{v\}$ and v. Hence $AD(vG) - \breve{}\{v\} \cong G^v$. □

Theorem 2.2. *Let G be a graph of order p. Let v be any vertex in G and v' be the anti-duplication vertex of v in $AD(vG) = G'$. Then*

(i) $[N[v]]^C = N'(v')$
(ii) $d'(v) = d(v)$

(iii) $N(v) = N'(v)$
(iv) $d'(v') = p - ˘1 - ˘d(v)$.

Proof. By the definition of G', the vertex v' is adjacent to all vertices which are non-adjacent to v in G. This implies that $[N[v]]^C = N'(v')$. Since the degree of v is unaltered in both G and G', we have $d(v) = d'(v)$ and $N(v) = N'(v)$. Now $d'(v') =\mid N'(v') \mid=\mid [N[v]]^C \mid= p - ˘1˘ - d(v)$. □

Corollary 2.1. *Let G be a graph of order p. If v' is an anti-duplication vertex of v in $AD(vG) = G'$, then $d'(v') + d(v) = p˘ - 1$.*

Proof. The result follows from Theorem 2.2 (iv). □

Corollary 2.2. *Let $G(p, q)$ be a graph and let v be a vertex of G. Let $G'(p', q')$ be the graph $AD(G)$. Then $p' = p + 1$ and $q' = p + q - ˘d(v) - ˘1$.*

Proof. By the definition of anti-duplication of the vertex v in G, G' has one vertex more than G and the new vertex v' has the degree same as $\mid [N[v]]^c \mid$. This implies that $p' = p+1$ and $q' = q+d'(v')$. By Theorem 2.2, $q' = p+q-˘d(v)-˘1$. This completes the proof of the theorem. □

Theorem 2.3. *Let G be a graph of order p and let u and v be any two non-adjacent vertices of G. Then u is the anti-duplicating vertex of v in $AD(vG) = G'$ if and only if $d_G(u) + d_G(v) = p - ˘2$ and $N_G(u) \cap N_G(v) = \phi$.*

Proof. Let G be a graph of order p and let u be the anti-duplication vertex of v in $G - ˘u$. Then by Corollary 2.1, $d_G(u) + d_{G-u}(v) = (p˘ - 1)˘ - 1 = p˘ - 2$. But $d_{G-u}(v) = d_G(v)$ and hence $d_G(u) + d_G(v) = p - ˘2$. Since u is the anti-duplication vertex of v in G, by Theorem 2.2, we have $N_G(u) = [N_{G-u}[v]]^c$ and $N_{G-u}(v) = N_G(v)$. Since $N_{G-u}(v) \cap [N_{G-u}[v]]^c = \phi$, we have $N_G(u) \cap N_G(v) = \phi$.

Conversely, let $d_G(u) + d_G(v) = p - ˘2$ and $N_G(u) \cap N_G(v) = \phi$. Let $d_G(v) = n$. Then $d_G(u) = p - ˘2 - ˘n$. Since uv is not an edge in G and $N_G(u) \cap N_G(v) = \phi$, each vertex w distinct from both u and v is adjacent to either u or v in G. Clearly u is adjacent to all the vertices which are non-adjacent to v in G. This implies that u is the anti-duplication vertex of v. Hence the proof. □

Theorem 2.4. *Let $G(p, q)$ be a connected graph and let v be a vertex of G. Let G' be the graph $AD(vG)$. Then G' is disconnected if and only if $d(v) = p˘ - 1$.*

Proof. Let G be a connected graph of order p. Let v' be the anti-duplication vertex of v in G'. Let G' be disconnected. Suppose that $d(v) < p - 1$. Then there exists at least one vertex non-adjacent to v in both G and G', say u. G is connected implies that there exists a $u - v$ path in G and hence in G'. Since v is non-adjacent to u in G, v' is adjacent to u in G'. Now the edge $v'u$ and the path $u - v$ form a $v' - v$ path in G'. Since G is connected and there exists a $v' - v$ path in G', G' is connected, which is a contradiction to G' is disconnected. Hence $d(v) = p - 1$.

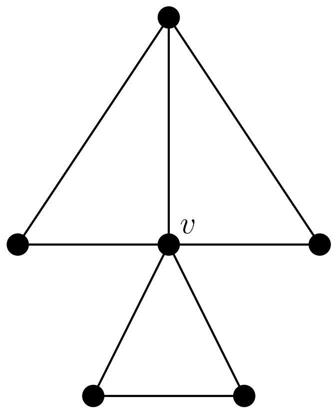

Fig 2.6. G

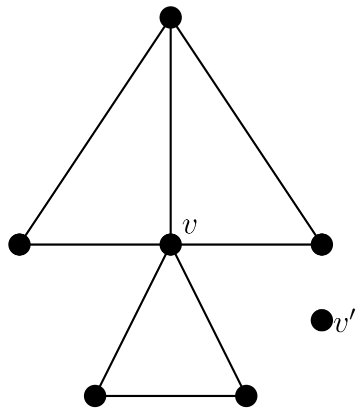

Fig 2.7. $AD(vG)$

Conversely, assume that $d(v) = p - 1$. This implies that v is adjacent to all vertices of G. By definition, v' is non-adjacent to all vertices of G'. Therefore v' is an isolated vertex in G'. This implies that G' is disconnected. Hence the theorem. □

Theorem 2.5. *Let $G(p, q)$ be a disconnected graph and let $AD(vG) = G'$ be the graph obtained by anti-duplicating the vertex v. Then G' is connected if and only if there exists at least one edge uw in G such that $u \in N(v)$ and $w \in [N[v]]^c$.*

Proof. Let A be a component of G containing the vertex v and B be the union of other components of G such that $G = A \cup B$. Assume that G' is connected. Suppose that $N_A(v) = \phi$. Then v is an isolated vertex in G and hence in G'. This is a contradiction to G' is connected and hence $N_A(v) \neq \phi$. Suppose that $[N_A[v]]^c = \phi$. Then v is adjacent with all the vertices of $A - v$ in G. By definition, v' is non-adjacent with all the vertices of $A - v$ in G'. Also v and v' are non-adjacent to each other in G'. This implies that A is a component of G' which is a contradiction to G' is connected and hence $[N_A[v]]^c \neq \phi$. Thus $N_A(v) \neq \phi$ and $[N_A[v]]^c \neq \phi$.

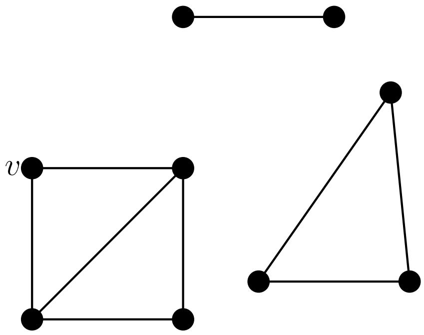

Fig 2.8. G

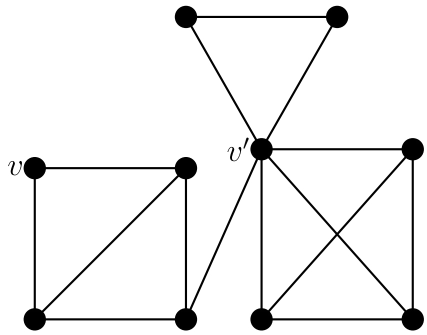

Fig 2.9. $AD(vG)$

Let $v, v_1, v_2, ..., v_n$ be the vertices of A in G. Without loss of generality, let us assume $N_A(v) = \{v_1, v_2, ..., v_r\}$ and $[N_A[v]]^c = \{v_{r+1}, v_{r+2}, ..., v_n\}$, where $1 \leq r < r+1 \leq n$. Let $v_i \in N[v]$ and $v_j \in [N[v]]^c, 1 \leq i \leq r, r+1 \leq j \leq n$. Then $vv_i \in E(A)$ and $vv_j \notin E(A)$. Suppose that there does not exist an edge v_iv_j for any $i, j, 1 \leq i \leq r, r+1 \leq j \leq n$. Then $v_j, r+1 \leq j \leq n$, is neither adjacent to v nor adjacent to v_i of A in G. This implies that $G[v_j], r+1 \leq j \leq n$, is a component of A which is a contradiction to A is connected. Hence there exists at least one edge v_iv_j in $G, 1 \leq i \leq r, r+1 \leq j \leq n$.

Conversely, assume that there exists at least one edge uw in G such that $u \in N(v)$ and $w \in [N[v]]^c$. $u \in N(v)$ implies that vu is an edge is G and hence in G'. Since v is non-adjacent with all vertices of $[N[v]]^c$, we have v' is adjacent with all vertices of $[N_G[v]]^c$ in G'. $w \in [N_G[v]]^c$ implies that $v'w$ is an edge in G'. Let x and y be any two vertices in G'.

Case 1. $x, y \in N_G(v)$.

This implies that xvy is a $x - y$ path in G and hence in G'.

Case 2. $x, y \in [N[v]]^c$.

This implies that $xv'y$ is the required $x - y$ path in G'

Case 3. $x \in N[v]$ and $y \in [N[v]]^c$.

We consider the following four subcases:

Case 3.1 $x = u$ and $y = w$.

Clearly $xy = uw$ is a edge in G'.

Case 3.2. $x = u$ and $y \neq w$.

$y \in [N[v]]^c$ implies that vy is not an edge in G and hence y is adjacent with v' in G'. Now the edges $uw = xw, wv'$ and $v'y$ form a $x - y$ path in G'.

Case 3.3. $x \neq u$ and $y = w$.
Since $x \in N[v], xv$ is an edge in G and hence in G'. Now the edges xv, vu and $uw = uy$ form a $x - y$ path in G'.

Case 3.4. $x \neq u$ and $y \neq w$. ‘ Since $x \in N[v], xv$ is an edge in G and G'. Also $y \in [N[v]]^c$ implies that vy is not an edge in G and hence y is adjacent to v' in G'. Now the edges xv, vu, uw, wv' and $v'y$ form a $x - y$ path in G'.

Thus in all the possible cases there exists a $x - y$ path in G' and hence G' is connected. This completes the proof. □

Theorem 2.6. *Let v be any vertex of a graph G of order p and $AD(vG)$ be the graph obtained by anti-duplicating the vertex v, then $AD(vG)$ is a regular graph of degree n if and only if p is odd, $d_G(v) = d(v_i) = n = (p-1)/2$ and $d(v_j) = n - \breve{}1$ where $v_i \in N(v)$ and $v_j \in [N[v]]^c$.*

Proof. Let $v_1, v_2, ..., v_r = v, ..., v_p$ be the vertices of $G, 1 \leq r \leq p$. Let v' be anti-duplicating vertex of v in $AD(vG)$. Let $d(v) = d_G(v)$ and $d'(v) = d_{AD(vG)}(v)$. Let $AD(vG)$ be a regular graph of degree n. Then $d'(v) = d'(v') = d(v) = n$. This implies that $|N_G[v]| = |[N[v]]^c| = n$ and hence $p = 2n + 1$ is odd and $n = (p - \breve{}1)/2$.
Let $v_i \in N_G[v]$. Then v' is non-adjacent to v_i in $AD(vG)$. Since $AD(vG)$ is a regular graph of degree $n, d(v_i) = n$. Let $v_j \in\in [N_G[v]]^c$. Then v_j is adjacent with v' in $AD(vG)$ which implies $d(v_j) = n - \breve{}1$.

Conversely, let G satisfy the conditions (i) to (iii). Then $AD(vG)$ is a regular graph of degree n with $p + 1$ vertices. □

Theorem 2.7. *Let G be a graph of order p and v' be the anti-duplication vertex of v in G. Then $AD(vG)$ is not eulerian if G is either of the following*

(i) *G is eulerian;*
(ii) *v is an end vertex of G;*
(iii) *G is a regular graph;*
(iv) *G has an isolated vertex;*
(v) *G has an end vertex which is adjacent with v.*

Proof. Let $v_1, v_2, ..., v_p$ be the vertices of graph G. Without loss of generality let $v_i = v$ for some $i, 1 \leq i \leq p$. Let v' be the anti-duplication vertex of v in $AD(vG) = G'$.

Let G be an eulerian graph. Then every vertex of G is of even degree. Let $u \in [N_G[v]]^c$ and $d(u) = k$. Clearly, k is even. By the definition of $AD(vG), u$ is adjacent to v' and hence $d'(u) = k + 1$ which is odd. This implies that G' is not eulerian.

Let v be an end vertex of G. Then by definition, v is also an end vertex in G'. This implies that G' has a vertex of degree 1 and hence G' is not eulerian.

Let G be a regular graph of degree r. If r is even, then G is eulerian and hence by (i) G' is not eulerian and if r is odd, then v has odd degree in G and also in G' and hence G' is not eulerian.

Let G has an isolated vertex. Then v is either an isolated vertex or not. If v is an isolated vertex, then by definition of anti–duplication v is also an isolated vertex in G' and hence G' is not eulerian. If v is not an isolated vertex, then let $v_j(\neq v)$ be an isolated vertex in G. Then v_j is adjacent to v' in G'. This implies that v_j has degree 1 in G' and hence G' is not eulerian.

Let u be an end vertex which is adjacent to v in G. Then $d(u) = 1$. By the definition of $AD(vG), v'$ is non-adjacent with u in G' and hence $d'(v) = 1$ in G', which is odd. Hence G' is not eulerian. This completes the proof. $\square$

3. Conclusion

In this paper, we introduced the concept of Anti-duplication of a vertex v in G and its properties are studied. We also characterized a graph to be connected, disconnected, regular and eulerian after anti-duplication of the vertex v in G.

References

[1] C. Jayasekaran: *Self vertex Switching of trees*, Ars Combinatoria, **127** (2016), 33–43.

[2] C. Jayasekaran, V. Prabavathy: *A characterisation of duplication self vertex switching in graphs*, International Journal of Pure and Applied Mathematics, **118**(2) (2018), 149–156.

[3] E. Sambathkumar: *On Duplicate Graphs*, Journal of Indian Math. Soc., **37** (1973), 285–293.

[4] F. Harrary: *Graph Theory*, Addition Wesley, 1972.

[5] J. H. LINT, J. J. SEIDEL: *Equilateral points in elliptic geometry*, In Proc. Kon. Nede. Acad. Wetensch., Ser. A, **69** (1966), 335–348.

[6] R. STANLEY: *Reconstruction from vertex switching*, J. Combinatorial theory. Series B, **38** (1985), 138–142.

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