SOLUTION OF NONLINEAR SECOND ORDER MULTI-POINT BOUNDARY VALUE PROBLEMS BY SHOOTING TYPE DIFFERENTIAL TRANSFORM ALGORITHM

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ABSTRACT. In this paper, Shooting Type Differential Transform Algorithm (STDTA), a modified version of Differential Transform Method (DTM) has been used to solve some nonlinear boundary value problems with multi-point boundary conditions. Using STDTA, the problems are solved and the solution is calculated in the form of a rapid convergent series. It demonstrates the efficiency and simplicity of the proposed method.

1. INTRODUCTION

Many linear and nonlinear boundary value problems occur in different areas of science and engineering. Various applications of these types of problems occur in fluid mechanics, quantum mechanics, optimal control, chemical reactor theory, aerodynamics, reaction-diffusion process, geophysics and other related fields of applied science [1].

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2. DIFFERENTIAL TRANSFORMATION METHOD

Differential transform method is a numerical method based on Taylor expansion. The method was first proposed by Zhou (1998) to solve linear and nonlinear differential equations with initial conditions in electrical circuit analysis [2]. Multi-point boundary value problems (MPBV) for ordinary differential equations arise in the mathematical modelling of viscoelastic and elastic flows, deformation of beams and plate deflection theory. Some basic definitions and results are given below:

Definition 2.1. *The one-dimensional differential transform of a function $y(x)$ at the point $x = x_0$ is defined as follows:*

$$(2.1) \qquad Y(k) = \frac{1}{k!} \left[\frac{d^k y(x)}{dx^k} \right] \Bigg|_{x=x_0}$$

where $y(x)$ is the original function and $Y(k)$ is the transformed function.

Definition 2.2. *The differential inverse transform of $Y(k)$ is defined as follows:*

$$(2.2) \qquad y(x) = \sum_{k=0}^{\infty} Y(k)(x - x_0)^k.$$

From (2) and (2.2) one gets

$$y(k) = \sum_{k=0}^{\infty} Y(k) = \frac{1}{k!} \left[\frac{d^k y(x)}{dx^k} \right] (x - x_0)^k .$$

The proofs of following theorems can be easily obtained from the corresponding Taylor series and algebra of power series [3].

Theorem 2.1. *If $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) \pm H(k)$.*

Theorem 2.2. *If $f(x) = \lambda g(x)$, then $F(k) = \lambda G(k)$ where λ is a constant.*

Theorem 2.3. *If $f(x) = g(x)h(x)$, then*

$$F(k) = \sum_{k_1=0}^{k} G(k_1)H(k - k_1).$$

Theorem 2.4. *If $f(x) = u(x)\dfrac{du(x)}{dx}$, then*

$$F(k) = \sum_{k_1=0}^{k} (k - k_1 + 1)U(k_1)U(k - k_1 + 1).$$

Theorem 2.5. *If* $f(x) = \dfrac{d^n u(x)}{dx^n}$, *then* $F(k) = \dfrac{(k+n)!}{k!} U(k+n)$.

Theorem 2.6. *If* $f(x) = \sin(\omega x + \alpha)$, *then*

$$F(k) = \frac{\omega^k}{k!} \sin\left(\frac{k\pi}{2} + \alpha\right).$$

Let $\mathcal{B}$ be a Banach space and consider the functional equation defined on the Banach space $\mathcal{B}$, $Ty = b$ where T is an operator from $\mathcal{B}$ to $\mathcal{B}$, b is a given function of $\mathcal{B}$, and for each satisfying the functional equation [4, 5] is the solution. Assume that the functional equation has a unique solution for each $b \in \mathcal{B}$.

The operator T consists of both linear and non-linear terms, the linear term is decomposed into $L_1 + L_2$, where L_1 is the invertible, highest order derivative and L_2 is the remainder of the linear operator.

Thus $T = L_1 + L_2 + N$ where N is a non-linear operator. Hence the functional equation becomes

$$L_1 y = b - L_2 y - Ny.$$

Taking the Differential Transform on both sides of the above equation, we get the transformed equation as

$$(2.3) \qquad Y(k+n) = \frac{F(k)}{(k+n)!},$$

where $F(k)$ is the differential transform of $f(x, y, y', y'', \cdots y^{(n-1)}) = b - L_2 y - Ny$.

Then transformed conditions given with the problem can be written as:

$$(2.4) \qquad Y(k) = J, Y(m) = \sum_{k=0}^{N} \prod_{i=1}^{m-1} (k-i) Y(k) = I_m, (m < n),$$

where m is the order of the derivative in the boundary conditions and J, I_m are real constants. Using equations (2.3) and (2.4) the values of $Y(i), i = 1, 2, 3, \cdots$ can determined and then using inverse differential transformation, the following approximate solution can be determined as:

$$(2.5) \qquad y_N = \sum_{k=0}^{N} Y(k) x^k.$$

Usually DTM is used for solving initial value problems. To solve boundary value problems efficiently the authors [6, 7] have introduced Shooting Type Differential Transform Algorithm (STDTA). The basic steps of STDTA are as follows:

(i) Converting the given boundary value problem into an initial value problem by assuming the missing initial conditions;

(If the differential equation is of order n and there are m conditions given at the initial point and the remaining $n - m$ conditions are given at other points, assumptions are made on the remaining $n - m$ initial conditions. In the case of second order boundary value problem one assumes $u'(0) = \alpha$.)

(ii) applying the DTM to the converted initial value problem;

(In the case of second order boundary value problem the assumed condition transforms to $U(1) = \alpha$.)

(iii) computing the coefficients $Y(k+n)$ for $k \geq 0$ using (2.3) up to a specified level; and

(i) finding the value(s) of the assumed condition(s) by applying the boundary condition(s) at the other point to the approximate solution (2.5).

(In the case of second order boundary value problem α, the only assumed constant is found out by applying the condition at the second point to the approximate solution).

For solving multi-point boundary value problems step (iv) can be slightly modified as applying the remaining conditions to the approximate solution to get the remaining assumed constants.

The effectiveness of STDTA is demonstrated here by applying it to some multi-point boundary value problems.

3. Illustrative Examples

Example 1. Consider the following three-point second order nonlinear ordinary differential equation by Geng [8]:

$$u'' + \frac{3}{8}u + \frac{2}{1089}[u']^2 + 1 = 0,$$

with the boundary conditions $u(0) = 0, \quad u(1) - u\left(\frac{1}{3}\right) = 0$.

Let

(3.1) $$u'' + \frac{3}{8}u + \frac{2}{1089}[u']^2 + 1 = 0,$$

$$u(0) = 0, \quad u(1) - u\left(\frac{1}{3}\right) = 0\,.$$

Taking the differential transform of (3.1), one obtains

$$U(k + 2) = \frac{1}{(k + 1)(k + 2)}$$

$$\left[\frac{-3}{8}U(k) - \frac{2}{1089}\sum_{r=0}^{k}(r + 1)(k - r + 1)U(r + 1)U(k - r + 1) - \delta(k - 0)\right],$$

In the modified approach, one assumes that $u'(0) = \alpha$.

Taking the differential transform of $u(0) = 0$ and $u'(0) = \alpha$ yield
$U(0) = 0$, $U(1) = \alpha$. Putting $0, 1, 2, 3, \cdots$, in the transformed equation, the series coefficients $U(2), U(3), \cdots$, can be obtained as

$$U(0) = 0, U(1) = \alpha, \quad U(2) = \frac{1}{2}\left[\frac{-2\alpha^2}{1089} - 1\right],$$

$$U(3) = \frac{1}{6}\left[\frac{4\alpha^3}{(1089)^2} - \frac{2\alpha^2}{1089} - \frac{3251\alpha}{8712}\right],$$

$$U(4) = \frac{1}{12}[6.316504219 * (10^{-3})\alpha^2 + 0.185663452 + 3.372905952 * (10^{-6})\alpha^3 -$$

$$1.238900258 * (10^{-8})\alpha^4], \cdots$$

$$U(7) = [2.679938467 * (10^{-18}) * \alpha^7 - 5.501947156 * (10^{-14}) * \alpha^6 -$$

$$-1.064084847 * (10^{-12}) * \alpha^5$$

$$+2.110474139 * (10^{-10}) * a^4 + 4.394001132 * (10^{-8}) * \alpha^3 -$$

$$-4.376252405 * (10^{-8}) * \alpha^2$$

$$-8.375046934 * (10^{-6}) * \alpha].$$

Then the successive approximations to the solution are obtained, using
$u_n(x) = \sum_{k=0}^{n} U(k)x^k$. The seventh approximation is:

$$u_7(x) = \alpha x - [9.182736455 * (10^{-4}) * \alpha^2 + 0.5]x^2 + [5.621509921 * (10^{-7}) * \alpha^3$$
$$- 3.060912152 * (10^{-4}) * \alpha^2 - 0.062193908 * \alpha]x^3 + [8.5666541 * (10^{-5}) * \alpha^2$$
$$+ 3.372905952 * (10^{-6}) * \alpha^3 - 1.238900258 * (10^{(-8)}) * \alpha^4 + 0.015471954]x^4$$
$$+ [1.327257699 * (10^{-12}) * \alpha^5 - 5.162084408 * (10^{-10}) * \alpha^4 - 4.59848158 * (10^{-7}) * \alpha^3$$
$$+ 5.739118458 * (10^{-6}) * \alpha^2 + 1.120503143 * (10^{-3}) * \alpha]x^5 + [-1.915234775 * (10^{-15}) * \alpha^6$$
$$+ 7.584329638 * (10^{-13}) * \alpha^5 + 7.542596149 * (10^{-10}) * \alpha^4 - 2.786659534 * (10^{-8}) * \alpha^3$$
$$- 9.125112654 * (10^{-6}) * \alpha^2 - 2.470403453 * (10^{-4})]x^6 + [2.679938467 * (10^{-18}) * \alpha^7$$
$$- 5.501947156 * (10^{-14}) * \alpha^6 - 1.064084847 * (10^{-12}) * \alpha^5 + 2.110474139 * (10^{-10}) * \alpha^4$$
$$+ 4.394001132 * (10^{-8}) * \alpha^3 - 4.376252405 * (10^{-8}) * \alpha^2 - 8.375046934 * (10^{-6}) * \alpha]x^7.$$

The ith approximation to the solution $u(x), u_i(x)$, is the terms up to x^i of the above expression. Now applying the condition $u(1) = u\left(\dfrac{1}{3}\right)$ to $u_n(x)$ yields the approximate values for α, namely α_n, for different values of n. They are tabulated in Table 1. From the table, it can be observed that the sequence a_n converges. Substituting these values of a_n in the corresponding $u_n(x)$, the nth approximation to the solution, $u(x)$ is obtained.

Table 1. Values of α_n

n	α_n
1	0
2	0.6670757
3	0.7332268
4	0.7077722
5	0.7064667
6	0.7068806
7	0.7068904

Table 2 gives the values of $u_n(x)$, evaluated at $x = 0.1, 0.2, 0.3, \cdots, 0.9, 1.0$, for different values of n. From this table, it is clear that $u_n(x)$ is convergent. Comparison with that of the earlier results are presented in Table 3. From the table the effectiveness of STDTA can be noted, since only up to x^7 terms in the series have been considered here, whereas in [9] the author had taken the terms up to x^{17}.

Table 2. Convergence of the sequence u_n

x	u_2	u_4	u_6	u_7
0.1	0.06170	0.0657	0.0656	0.0656
0.2	0.1134	0.1212	0.1210	0.1210
0.3	0.1552	0.1662	0.1659	0.1659
0.4	0.1868	0.2006	0.2002	0.2002
0.5	0.2086	0.2242	0.2238	0.2238
0.6	0.2204	0.2369	0.2365	0.2365
0.7	0.2222	0.2388	0.2383	0.2383
0.8	0.2139	0.2297	0.2293	0.2293
0.9	0.1956	0.2097	0.2093	0.2093
1.0	0.1667	0.2073	0.2079	0.2079

Table 3. Comparison with the Existing Results

x	STDTA	ADM [7]	SIM [7]	RKM [6]
0.1	0.0656	0.0656	0.0656	0.0656
0.2	0.1210	0.1209	0.1211	0.1209
0.3	0.1659	0.1658	0.1661	0.1658
0.4	0.2002	0.2001	0.2004	0.2001
0.5	0.2238	0.2236	0.2240	0.2236
0.6	0.2365	0.2363	0.2367	0.2363
0.7	0.2383	0.2382	0.2385	0.2382
0.8	0.2293	0.2291	0.2295	0.2291
0.9	0.2093	0.2091	0.2095	0.2091
1.0	0.2079	0.2000	0.1895	0.1891

Example 2. Consider the nonlinear multi-point boundary value problem

$$u''(x) + u(x)u'(x) = (\cos x - 1)\sin x \,,$$

$$u(0) = 0, u(1) = \sum_{i=0}^{4} \frac{1}{1+i} u\left(\frac{i}{5}\right) + 0.3277 \,.$$

Let

(3.2) $$u''(x) + u(x)u'(x) = \frac{1}{2}\sin 2x - \sin x \,,$$

$$u(0) = 0, u(1) = \sum_{i=0}^{4} \frac{1}{(1+i)} u\left(\frac{i}{5}\right) + 0.3277 \,.$$

Taking the differential transform of (3.2), one obtains

$$U(k+2) = \frac{1}{(k+1)(k+2)} \left[- \left(\sum_{r=0}^{k} (k-r+1)U(r)U(k-r+1) \right) + \frac{1}{2} \left(\frac{2^k}{k!} \sin \frac{k\pi}{2} \right) - \frac{1}{k!} \cdot \sin \left(\frac{k\pi}{2} \right) \right]$$

In the modified approach, one assumes that $u'(0) = \alpha$.

Taking the differential transform of $u(0) = 0$ and $u'(0) = \alpha$, yields $U(0) = 0, U(1) = \alpha$. Putting $k = 0, 1, 2, 3, \cdots$, in the transformed equation, the series coefficients $U(2), U(3), \cdots$, can be obtained as:

$$U(0) = 0, U(1) = \alpha, \quad U(2) = 0, \quad U(3) = \frac{-\alpha^2}{6}, \quad U(4) = 0,$$

$$U(5) = \frac{1}{20} \left[\frac{2\alpha^3}{3} - \frac{1}{2} \right], \quad U(6) = 0,$$

$$U(7) = \frac{1}{42} \left[\frac{-17\alpha^4}{60} + \frac{3\alpha}{20} + \frac{1}{8} \right].$$

Then the successive approximations to the solutions are obtained, using $u_n(x) = \sum_{k=0}^{n} U(k)x^k$. The seventh approximation is:

$$u_7(x) = \alpha x - \frac{\alpha^3}{6}x^3 + \frac{1}{20} \left[\frac{2\alpha^3}{3} - \frac{1}{2} \right] x^5 + \frac{1}{42} \left[\frac{-17\alpha^4}{60} + \frac{3\alpha}{2} + \frac{1}{8} \right] x^7.$$

The ith approximation to the solution $u(x), u_i(x)$, is the terms upto x^i of the above expression.

Now applying the condition $u(1) = \sum_{i=0}^{4} u \left(\frac{i}{5} \right) + 0.3277$ to $u_n(x)$, the approximate values for α, namely α_n, for different values of n, are obtained. They are tabulated in Table 4. From the table it is clear that the sequence α_n converges. Substituting these values of α_n in the corresponding $u_n(x)$, the nth approximation to the solution, $u(x)$ is obtained.

Table 5 gives the values of $u_n(x)$, evaluated at $x = 0.1, 0.2, 0.3, \cdots, 0.9, 1.0$, for different values of n. From this table, it is clear that $u_n(x)$ is convergent. In Table 6 the results obtained here, are compared with the Homotopy Perturbation Method [10] and the Exact Solution. It can be observed from the table that our

solution matches better with the exact solution, as the values of x increase, than by HPM.

Table 4. Values of α_n

n	a_n
1	0.71759124
3	1.04157923
5	0.99932998
7	1.00537851

Table 5. Convergence of the sequence $u_n(x)$

x	u_3	u_5	u_7
0.1	0.103977	0.099767	0.100369
0.2	0.206869	0.198537	0.199728
0.3	0.307592	0.295325	0.297093
0.4	0.405059	0.389164	0.391513
0.5	0.498188	0.479118	0.482158
0.6	0.585892	0.564289	0.568424
0.7	0.667086	0.643830	0.650108
0.8	0.740686	0.716953	0.727666
0.9	0.805608	0.782941	0.802571
1.0	0.860764	0.841153	0.852044

Table 6. Comparison with the Existing results

x	HPM [8]	STDTA	Exact Solution [8]
0.1	0.988379	0.100369	0.099833
0.2	0.983472	0.199728	0.198669
0.3	0.978774	0.297093	0.295520
0.4	0.978006	0.391513	0.389418
0.5	0.972140	0.482158	0.479426
0.6	0.970688	0.568424	0.564642
0.7	0.970636	0.650108	0.644215
0.8	0.971932	0.727666	0.717356
0.9	0.974389	0.802571	0.783327
1.0	0.977704	0.852044	0.841471

CONCLUSION

In this paper the Shooting Type Differential Transform Algorithm (STDTA) is successfully applied to solve nonlinear multi-point boundary value problems. The study shows that the method leads to more reliable results with less computational work.

REFERENCES

[1] A. A. OPANUGA, S. O. EDEKI, H. I. OKAGBUE, G. O. AKINLABI: *Numerical Solution of Two-point boundary value problems via Differential Transform Method*, Global Journal of Pure and Applied Mathematics, **11**(2) (2015), 801–806.

[2] J. K. ZHOU: *Differential Transformation and its Applications for Electrical Circuits*, Huazhong University Press, Wuhan, China, 1986.

[3] M. M. RASHIDI, A. J. CHAMKHA, M. KEIMANESH: *Application of Multi-Step Differential Transform Method on Flow of a Second-Grade Fluid over a Stretching or Shrinking Sheet*, American Journal of Computational Mathematics, **6** (2011), 119–128.

[4] C. LILI, M. CUI: *The exact solution for solving a class of nonlinear operator equation in reproducing kernel space*, Appl. Math. Comput., **143**(2-3) (2003), 15–27.

[5] V. DAFTARDAR-GAJJI, H. JAFARI: *An iterative method for solving nonlinear functional equation*, J. Math. Anal. Appl., **316** (2006), 753-763.

[6] T. R. SIVAKUMAR, S. GEORGE: *Application of STDTA for Nonlinear differential equations with one boundary conditions at Infinity*, Asian Journal of Applied sciences, **2** (2019), 1-6.

[7] T. R. SIVAKUMAR, S. GEORGE: *Application of Differential Transform Method to Nonlinear Boundary Value Problems*, in Proceedings of ICNumACA, **18** (2019), 212–219.

[8] F. Z. GENG: *A numerical algorithm for nonlinear multi-point boundary value problems*, Journal of Computational and Applied Mathematics, **236** (2012), 1789–1794.

[9] M. TATARI, M. DEHGHAN: *The use of the Adomian decomposition method for solving multipoint boundary value problems*, Physica Scripta, **73** (2006), 672–676.

[10] S. DAS, S. KUMAR, O. P. SINGH: *Solutions of Nonlinear Second Order Multi-point Boundary Value Problems by Homotopy Perturbation Method*, International Journal of Applications and Applied Mathematics, **5**(10) (2010), 1592–1600.

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