

## ON NUMERICAL RANGE AND NUMERICAL RADIUS OF A SPECIAL PAIR OPERATOR MATRICES

NIRANJAN BORA

**ABSTRACT.** This paper considers Linear two-parameter eigenvalue problems in terms of matrix operators. Generally, for spectral analysis two-parameter problem is reduced into a system of generalized eigenvalue problems using a special pair of determinant operator matrices on tensor product space. In this work, some inequalities on numerical range and numerical radius of this special pair of operator matrices arising from two-parameter problem will be derived.

### 1. INTRODUCTION

In 1932 Marshall Stone [11] first coined about numerical range of a bounded linear operator  $T$  over the complex Hilbert space  $H$  and is defined as

$$W(T) := \{ \langle Tx, x \rangle : x \in H, \|x\| = 1 \}$$

where  $\langle \cdot, \cdot \rangle$  denotes standard inner product and  $\|x\| = \sqrt{\langle x, x \rangle}$  is the induced norm. Confining this definition for finite dimensional case and we consider numerical range of any  $n \times n$  matrix  $A$  over  $\mathbb{C}$ , i.e,

$$W(A) := \{ x^* Ax : x \in \mathbb{C}, \|x\| = 1 \}$$

where  $\|x\| = \sqrt{x^* x}$  is the Euclidean length  $x \in \mathbb{C}^n$  and  $x^*$  is the transpose conjugate of  $x$ . The quantity  $W(A)$  is useful to locate eigenvalues, to obtain norm bounds, to deduce double algebraic and analytic properties of matrices

---

2010 *Mathematics Subject Classification.* 15A18, 47A12.

*Key words and phrases.* Numerical range, numerical radius, two-parameter eigenvalue problem.



and it also help to find dilutions with simple structure. It follows from celebrated Toeplitz-Hausdorff Theorem that  $W(A)$  is convex and compact subset of the complex plane [14]. Moreover, the spectrum of  $A$  are always lies in  $W(A)$  [13]. Numerical radius of  $A$  is denoted by  $w(A)$  and is defined as

$$w(A) := \text{Max} \{ |z| : z \in W(A) \} .$$

Researcher have studied extensively on numerical range and numerical radius for operators and matrices over the years. Various results of numerical range and numerical radius for matrices have been found in [6, 7, 9, 12, 14, 16], and the references therein. The rest of the paper is organized as follows: In Section 2 an abstract formulation of linear two-parameter eigenvalue problem is presented. In Section 3, some inequality of numerical radius of certain operator matrices are derived. Section 4 contains concluding remarks.

## 2. LINEAR TWO-PARAMETER MATRIX EIGENVALUE PROBLEMS

Linear two-parameter eigenvalue problem [4] considered here is

$$(2.1) \quad \begin{aligned} \mathbb{W}_1(\lambda_1, \lambda_2)x_1 &:= (B_{01} - \lambda_1 B_{11} - \lambda_2 B_{12})x_1 = 0 \\ \mathbb{W}_2(\lambda_1, \lambda_2)x_2 &:= (B_{02} - \lambda_1 B_{21} - \lambda_2 B_{22})x_2 = 0 \end{aligned}$$

where  $\lambda_1, \lambda_2 \in \mathbb{C}$  are spectral parameters;  $x_i \in \mathbb{C}^{n_i}$ ; and  $B_{ij}$  are  $n_1 \times n_2$  over  $\mathbb{C}$ ;  $i := 0 : 3$ ;  $j := 1 : 2$ . Denote the problem (2.1) by  $\mathbb{W}$ . If for some  $\lambda_1, \lambda_2$  the problem  $\mathbb{W}$  has a solution  $x_i \neq 0$ ;  $i := 1 : 2$ , then the pair  $(\lambda_1, \lambda_2)$  is called eigenvalue and the corresponding tensor product  $x = x_1 \otimes x_2$  is called the eigenvector, where  $\otimes$  stands for Kronecker product. An extensive analysis of the spectral theory of  $\mathbb{W}$  and its related classical results can be found in the books [2, 4] and in the papers [8, 10, 15]. The standard method to study the spectrum of  $\mathbb{W}$  by transforming it into a commuting pair of operators matrices by considering the following operator determinants

$$\begin{aligned} \Delta_0 &:= B_{11} \otimes B_{22} - B_{12} \otimes B_{21} \\ \Delta_1 &:= B_{01} \otimes B_{22} - B_{12} \otimes B_{02}; \quad \Delta_2 := B_{11} \otimes B_{02} - B_{01} \otimes B_{21} . \end{aligned}$$

Generally, for spectral analysis the problem  $\mathbb{W}$  is considered as nonsingular i.e when  $\Delta_0$  is nonsingular. A nonsingular system (2.1) can be transformed into a



system of joint generalized eigenvalue problems [4] of the form

$$(2.2) \quad \Delta_i x = \lambda_i \Delta_0 x, \quad i := 1 : 2.$$

Denote  $\Gamma_i := \Delta_0^{-1} \Delta_i$ ;  $i := 1 : 2$ . For nonsingular  $\mathbb{W}$  the operator matrices  $\Gamma_i$  commute for  $i := 1 : 2$  and all the eigenvalues of (2.1) are also the eigenvalues of (2.2).

### 3. MAIN RESULTS

The numerical ranges and numerical radii for  $\Gamma_i$ ,  $i := 1 : 2$  becomes,

$$\begin{aligned} W(\Gamma_i) : &= \{x^* \Gamma_i x : x \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}, x^* x = 1\}, \\ w(\Gamma_i) : &= \text{Max} \{|z| : z \in W(\Gamma_i)\}. \end{aligned}$$

Moreover for  $i := 0 : 3$ ;  $j = 1 : 2$ ,

$$W(\Delta_i) := \{x^* \Delta_i x : x \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}, x^* x = 1\}, \quad W(B_{ij}) := \{x_i^* B_{ij} x_i : x_i \in \mathbb{C}^{n_i}, x_i^* x_i = 1\}.$$

Here some results on  $W(\Gamma_i)$  and  $w(\Gamma_i)$  will be presented for  $i := 1 : 2$ .

**Theorem 3.1.** *Let  $\|x_i\| = 1$  for  $i := 1 : 2$ . Then for the operator matrices  $\Delta_i$ , the following results hold:*

- (i)  $W(\Delta_0) = W(B_{11})W(B_{22}) - W(B_{12})W(B_{21})$ ,
- (ii)  $W(\Delta_1) = W(B_{01})W(B_{22}) - W(B_{12})W(B_{02})$ ,
- (iii)  $W(\Delta_2) = W(B_{11})W(B_{02}) - W(B_{01})W(B_{21})$ .

*Proof.* Let  $\|x_i\| = 1$  for  $i := 1 : 2$ . Then numerical range of  $\Delta_0$  becomes,

$$\begin{aligned} W(\Delta_0) &= x^*(B_{11} \otimes B_{22} - B_{12} \otimes B_{21})x = x^*(B_{11} \otimes B_{22})x - x^*(B_{12} \otimes B_{21})x \\ &= (x_1^* B_{11} x_1) \otimes (x_2^* B_{22} x_2) - (x_1^* B_{12} x_1) \otimes (x_2^* B_{21} x_2). \end{aligned}$$

Since  $x_i^* B_{ij} x_i$ ,  $i := 1 : 2$  are scalars, so kronecker product appear in above equation becomes ordinary multiplication. Thus

$$\begin{aligned} x^*(B_{11} \otimes B_{22} - B_{12} \otimes B_{21})x &= x^*(B_{11} \otimes B_{22})x - x^*(B_{12} \otimes B_{21})x \\ &= (x_1^* B_{11} x_1)(x_2^* B_{22} x_2) - (x_1^* B_{12} x_1)(x_2^* B_{21} x_2) = W(B_{11})W(B_{22}) - W(B_{12})W(B_{21}). \end{aligned}$$

Second and third equation can be derived in a similar fashion.  $\square$

**Theorem 3.2.** *Let  $K_{\Delta_0}$  is the condition number of the operator matrix  $\Delta_0$ . Then*

$$w(\Gamma_i) \leq K_{\Delta_0} \frac{\|\Delta_i\|}{\|\Delta_0\|}; \quad \text{for } i := 1 : 2$$



*Proof.* It follows from Cauchy-Schwarz inequality that  $w(A)$  is bounded for any matrix  $A$ . i.e.,  $w(A) \leq \|A\|$ . Replacing  $\Gamma_i$  in place of  $A$  we get

$$w(\Gamma_i) \leq \|\Delta_0^{-1} \Delta_i\| \leq \|\Delta_0^{-1}\| \|\Delta_i\| = \|\Delta_0^{-1}\| \|\Delta_0\| \|\Delta_i\| / \|\Delta_0\| = K_{\Delta_0} \frac{\|\Delta_i\|}{\|\Delta_0\|}.$$

□

The following estimate [9] for  $w(\Gamma_i)$ ,  $i := 1 : 2$  is automatically true.

$$(3.1) \quad w(\Gamma_i) = w(\Delta_0^{-1} \Delta_i) \leq 4w(\Delta_0^{-1})w(\Delta_i).$$

But this estimate is not computationally friendly due to the involvement of matrix operators with Kronecker structure of high dimension. If  $\Delta_0$  and  $\Delta_i$  commute for  $i := 1 : 2$ , then we have

$$\Delta_i \Delta_0 = \Delta_0 \Delta_i \Rightarrow \Delta_0^{-1} \Delta_i = \Delta_i \Delta_0^{-1}$$

and it follows from Theorem 6.5 (a) [3] that

$$w(\Gamma_i) = w(\Delta_0^{-1} \Delta_i) \leq 2w(\Delta_0^{-1})w(\Delta_i).$$

**Theorem 3.3.** (Theorem 6.8, [3]) Let  $A$  and  $B$  be any two square matrices. Then

$$w(A \otimes B) \leq \text{Min} \{w(A) \|B\|, \|A\| w(B)\} \leq 4w(A)w(B).$$

**Theorem 3.4.** Let  $\Delta_0$  and  $\Delta_i$  commute for  $i := 1 : 2$ . Then

$$(i) \quad w(\Gamma_1) \leq 2(\|B_{01}\| \|B_{22}\| + \|B_{12}\| \|B_{02}\|)w(\Delta_0^{-1}).$$

$$(ii) \quad w(\Gamma_2) \leq 2(\|B_{11}\| \|B_{02}\| + \|B_{01}\| \|B_{21}\|)w(\Delta_0^{-1}).$$

*Proof.* Let  $\Delta_0$  and  $\Delta_1$  commute, i.e.,  $\Delta_i \Delta_0 = \Delta_0 \Delta_i$ . Then it follows that  $\Delta_0^{-1} \Delta_i = \Delta_i \Delta_0^{-1}$ . Thus  $w(\Gamma_i) = w(\Delta_0^{-1} \Delta_i) = w(\Delta_i \Delta_0^{-1})$ . For any two operator [1] is well known that  $w(AB) \leq 2\|A\|w(B)$ . Now

$$\begin{aligned} w(\Gamma_1) &= w(\Delta_1 \Delta_0^{-1}) \leq 2\|\Delta_1\|w(\Delta_0^{-1}) \\ &\Rightarrow w(\Gamma_1) \leq 2\|B_{01} \otimes B_{22} - B_{12} \otimes B_{02}\|w(\Delta_0^{-1}) \\ &\leq 2(\|B_{01} \otimes B_{22}\| + \|B_{12} \otimes B_{02}\|)w(\Delta_0^{-1}) \\ &\leq 2(\|B_{01}\| \|B_{22}\| + \|B_{12}\| \|B_{02}\|)w(\Delta_0^{-1}) \end{aligned}$$

which proves the first inequality. Second inequality can be proved in a similar way. □

The above estimates can be refined more by imposing additional condition

**Theorem 3.5.** Let  $\Delta_0$  and  $\Delta_i$  commute for  $i := 1 : 2$  and  $\Delta_0$  be such that  $0 \notin W(\Delta_0^{-1})$ . Then



- (i)  $w(\Gamma_1) \leq \sqrt{3}(\|B_{01}\| \|B_{22}\| + \|B_{12}\| \|B_{02}\|)w(\Delta_0^{-1})$ .  
(ii)  $w(\Gamma_2) \leq \sqrt{3}(\|B_{11}\| \|B_{02}\| + \|B_{01}\| \|B_{21}\|)w(\Delta_0^{-1})$ .

*Proof.* To proof the first equation, let  $\Delta_0$  and  $\Delta_1$  commute. Then it follows that  $w(\Gamma_i) = w(\Delta_0^{-1}\Delta_i) = w(\Delta_i\Delta_0^{-1})$ .

Using (Corollary 1, [1]) we have

$$w(\Gamma_1) \leq \sqrt{3}\|\Delta_1\|w(\Delta_0^{-1}) \\ \Rightarrow w(\Gamma_1) \leq \sqrt{3}(\|B_{01}\| \|B_{22}\| + \|B_{12}\| \|B_{02}\|)w(\Delta_0^{-1}).$$

Second equation can be proved in a similar way.  $\square$

**Theorem 3.6.**  $w^2(\Gamma_i) \leq \|\Delta_i^*\| \|(\Delta_0^{-1})^*\| \|\Delta_0^{-1}\| \|\Delta_i\|$  for  $i := 1 : 2$ .

*Proof.* Using Kittaneh (Theorem 1, [5]) result on upper bound of numerical radius for any operator  $T$ , we have  $w^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|$ . Now,

$$w^2(\Gamma_i) \leq \frac{1}{2}\|(\Delta_0^{-1}\Delta_i)^*\Delta_0^{-1}\Delta_i + \Delta_0^{-1}\Delta_i(\Delta_0^{-1}\Delta_i)^*\| \\ \Rightarrow w^2(\Gamma_i) \leq \frac{1}{2}\|\Delta_i^*(\Delta_0^{-1})^*\Delta_0^{-1}\Delta_i + \Delta_0^{-1}\Delta_i\Delta_i^*(\Delta_0^{-1})^*\| \\ \Rightarrow w^2(\Gamma_i) \leq \frac{1}{2}(\|\Delta_i^*(\Delta_0^{-1})^*\Delta_0^{-1}\Delta_i\| + \|\Delta_0^{-1}\Delta_i\Delta_i^*(\Delta_0^{-1})^*\|) \\ \Rightarrow w^2(\Gamma_i) \leq \frac{1}{2}(2\|\Delta_i^*\| \|(\Delta_0^{-1})^*\| \|\Delta_0^{-1}\| \|\Delta_i\|) \\ \Rightarrow w^2(\Gamma_i) \leq \|\Delta_i^*\| \|(\Delta_0^{-1})^*\| \|\Delta_0^{-1}\| \|\Delta_i\|. \quad \square$$

**Theorem 3.7.** For all  $i := 1 : 2$ , the following estimates of  $w(\Gamma_i)$  are true:

(i)

$$w(\Gamma_1) \leq 4 \quad w(\Delta_0^{-1})(w(\text{Min}\{w(B_{01})\|B_{22}\|, \|B_{01}\|w(B_{22})\} \\ + \text{Min}\{w(B_{12})\|B_{02}\|, \|B_{12}\|w(B_{02})\})) \\ \leq 16 \quad w(\Delta_0^{-1})(w(B_{01})w(B_{22}) + w(B_{12})w(B_{02}))$$

(ii)

$$w(\Gamma_2) \leq 4 \quad w(\Delta_0^{-1})(w(\text{Min}\{w(B_{11})\|B_{02}\|, \|B_{11}\|w(B_{02})\} \\ + \text{Min}\{w(B_{01})\|B_{21}\|, \|B_{01}\|w(B_{21})\})) \\ \leq 16 \quad w(\Delta_0^{-1})(w(B_{11})w(B_{02}) + w(B_{01})w(B_{21}))$$

*Proof.* Using (3.1) we get

$$w(\Gamma_1) \leq 4w(\Delta_0^{-1})w(\Delta_1) \\ \Rightarrow w(\Gamma_1) \leq 4w(\Delta_0^{-1})(w(B_{01} \otimes B_{22}) + w(B_{12} \otimes B_{02}))$$

Using Theorem 3.3, we have

$$w(\Gamma_1) \leq 4w(\Delta_0^{-1})(w(\text{Min}\{w(B_{01})\|B_{22}\|, \|B_{01}\|w(B_{22})\}$$



$$\begin{aligned}
& + \text{Min} \{w(B_{12}) \|B_{02}\|, \|B_{12}\| w(B_{02})\} \\
& \leq 16w(\Delta_0^{-1}) (w(B_{01})w(B_{22}) + w(B_{12})w(B_{02}))
\end{aligned}$$

Similarly, second inequality can be derived.  $\square$

#### 4. CONCLUDING REMARKS

A unified framework for numerical range and numerical radius of operator matrices arising from linear two-parameter eigenvalue problems is discussed. Some upper bounds of numerical radii have also been estimated.

#### REFERENCES

- [1] A. ABU-OMAR: *Numerical radius inequalities for products Hilbert Space operator II*, Numer. Funct. Anal. Opt., **41** (2) (2019), 127–133.
- [2] B. D. SLEEMAN: *Multiparameter spectral theory in Hilbert Space (Research Notes in Mathematics vol 22)*, Pitman, London, 1978.
- [3] C. K. LI: *Lecture notes on Numerical Range*, 2005.
- [4] F. V. ATKINSON: *Multiparameter eigenvalue problems, Vol. I, (Matrices and Compact Operators)*, Academic Press, New York, 1972.
- [5] F. KITTANEH: *Numerical radius inequalities for Hilbert Space operators*, Studia Math., **168**(1) (2005), 73–80.
- [6] H. GUELFEN, F. KITTANEH: *On Numerical radius inequalities for operator matrices*, Numer. Funct. Anal. Opt., **40**(1) (2019), 1231–1241.
- [7] H. L. GAU, Y. H. LU: *Extremality of Numerical radii of tensor product of matrices*, Linear Algebra Appl., **565** (2019), 82–98.
- [8] H. VOLKMER: *Multiparameter eigenvalue problems and expansion theorem*, Lect. Note. Math. 1356, Springer-Verlag, Berlin, New York, 1988.
- [9] M. AL-DOLAT, K. AL-ZOUBI, M. ALI, F. BANI-AHMAD: *General Numerical radius inequalities for matrices of operators*, Open Math., **14** (2016), 109–117.
- [10] M. E. HOCHSTENBACH, B. PLESTENJAK: *Backward error, condition numbers, and pseudospectrum for the Multiparameter eigenvalue problem*, Linear Algebra Appl., **375** (2003), 63–81.
- [11] M. STONE: *Linear transformations in Hilbert space*, American Mathematical Society, Prov., 1932.
- [12] P. BHUNIA, S. BAG, K. PAUL: *Numerical radius inequalities of operators matrices with applications*, Linear Mult. Algebra, **15** (2019), 1–10.
- [13] P. J. PSARRAKOS, M. J. TSATSOMEROS: *Numerical range: (in) a matrix nutshell*, Vol. 155 of Mathematics notes, Washington State University, 2002.



- [14] S. A. BROUGHTON, R. G. LAUTZENHEISER, T. WERNE: *The barycenter of Numerical range of a matrix*, Math. Sc. Tech. Report. Rose-Hulman Institute of Technology, 2007.
- [15] T. KOŠIR: *Finite-dimensional Multiparameter Spectral Theory: The nonderogatory Case*, Linear Algebra Appl., **212** (1994), 45–70.
- [16] W. BANI-DOMI, F. KITTANEH: *On Numerical radius inequalities for operator matrices*, Linear Mult. Algebra, **57**(4) (2009), 421–427.

DEPARTMENT OF MATHEMATICS

DIBRUGARH UNIVERSITY INSTITUTE OF ENGINEERING AND TECHNOLOGY

DIBRUGARH UNIVERSITY, ASSAM, INDIA

*E-mail address:* niranjanbora11@gmail.com