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VAGUE BI-INTERIOR IDEALS OF A Γ-SEMIRING

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ABSTRACT. In this paper, we introduce and study the concept of vague bi-interior ideal of a Γ-semiring as a generalization of vague bi-ideal and vague interior ideal and we characterize the vague bi-interior ideal of Γ-semiring to the crisp bi-interior ideals of Γ-semiring.

1. INTRODUCTION

In 1965, Zadeh, L.A. [18] introduced the study of fuzzy sets. Mathematically a fuzzy set on a set X is a mapping μ into the interval $[0,1]$; for x in X, $\mu(x)$ is called the membership of x belonging to X. This membership function gives only an approximation for belonging but it does not give any information of not belonging. To counter this problem and obtain a better estimation and analysis of data decision making, Gau, W.L. and Buehrer, D.J. [14] have initiated the study of vague sets with the hope that they form a better tool to understand, interpret and solve real life problems.

Further in 1995, Murali Krishna Rao, M. [15] introduced the concept of Γ-semiring which is a generalization of Γ-ring, ternary semiring and semiring and after that he introduced and studied the ideals of a Γ-semiring. Ideals play an important role in advance studies and uses of algebraic structures. Generalization of ideals in algebraic structures is necessary for further study of algebraic

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structures. Many mathematicians proved important results and characterization of algebraic structures by using the concept and the properties of generalization of ideals in algebraic structures. Murali Krishna Rao, M., [16, 17] introduced the concept of left(resp. right) bi-quasi ideal, bi-interior ideal a Γ-semiring and studied the properties of left bi-quasi ideals. However Bhargavi, Y. and Eswarlal, T. [1–11], [13] were developed the theory of vague sets on Γ-semirings. This paper is a sequel to our study. In this paper, we introduce and study the concept of vague bi-interior ideal of a Γ-semiring as a generalization of vague bi-ideal and vague interior ideal and we characterize the vague bi-interior ideal of Γ-semiring to the crisp bi-interior ideals of Γ-semiring.

2. Preliminaries

In this section we recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. *[15] Let E and Γ be two additive commutative semigroups. Then R is called Γ-semiring if there exists a mapping $R\times\Gamma\times R \to R$ image to be denoted by $a\alpha b$ if it satisfies the following conditions: For all $a, b, c \in R; \alpha, \beta \in \Gamma$.*

(ΓSR1) $a\alpha(b + c) = a\alpha b + a\alpha c$;
(ΓSR2) $(a + b)\alpha c = a\alpha c + b\alpha c$;
(ΓSR3) $a(\alpha + \beta)b = a\alpha b + a\beta b$;
(ΓSR4) $a\alpha(b\beta c) = (a\alpha b)\beta c$.

Definition 2.2. *[17] A non-empty subset B of a Γ-semiring R is said to be bi-interior ideal of R if B is a Γ-subsemiring of R and $R\Gamma B\Gamma R \cap B\Gamma R\Gamma B \subseteq B$.*

Definition 2.3. *[14] A vague set A in the universe of discourse X is a pair (t_A, f_A), where $t_A : X \to [0, 1]$, $f_A : X \to [0, 1]$ are mappings such that $t_A(x) + f_A(x) \leq 1$, for all $x \in X$. The functions t_A and f_A are called true membership function and false membership function respectively.*

Definition 2.4. *[14] The interval $[t_A(x), 1 - f_A(x)]$ is called the vague value of x in A and it is denoted by $V_A(x)$ i.e., $V_A(x) = [t_A(x), 1 - f_A(x)]$.*

Definition 2.5. *[14] Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be two vague sets of a universe of discourse X.*

The intersection of A *and* B *is defined as* $A \cap B = (t_{A\cap B}, f_{A\cap B})$*, where* $t_{A\cap B} = min\{t_A, t_B\}$ *and* $f_{A\cap B} = max\{f_A, f_B\}$*.*

The union of A *and* B *is defined as* $A \cup B = (t_{A\cup B}, f_{A\cup B})$*, where* $t_{A\cup B} = max\{t_A, t_B\}$ *and* $f_{A\cup B} = min\{f_A, f_B\}$*.*

The product $A\Gamma B$ *of* A *and* B *is defined as*

$$V_{A\Gamma B}(x) = \begin{cases} sup\{min\{V_A(y), V_B(z)\}/\ x = y\gamma z, where\ y, z \in R;\ \gamma \in \Gamma\} \\ [0,0], if\ for\ any\ y, z \in R; \gamma \in \Gamma,\ y\gamma z \neq x\,. \end{cases}$$

A vague set A *is contained in another vague set* B*,* $A \subseteq B$ *if and only if* $V_A(x) \leq V_B(x)$ *i.e.,* $t_A(x) \leq t_B(x)$ *and* $1 - f_A(x) \leq 1 - f_B(x)$*,* $\forall\ x \in X$*.*

Definition 2.6. *[14] Let* $A = (t_A, f_A)$ *be a vague set of a universe of discourse* X*. For* $\alpha, \beta \in [0,1]$ *with* $\alpha \leq \beta$*, the* (α, β)*- cut or vague cut of* A *is the crisp subset of* X *is given by* $A_{(\alpha,\beta)} = \{x \in X/\ V_A(x) \geq [\alpha, \beta]\}$ *i.e.,* $A_{(\alpha,\beta)} = \{x \in X/\ t_A(x) \geq \alpha\ and\ 1 - f_A(x) \geq \beta\}$*.*

Definition 2.7. *[14] For any subset* S *of a* Γ*-semiring* R *the vague characteristic set of* S *is a vague set* $\delta_S = (t_{\delta_S}, f_{\delta_S})$ *given by*

$$V_{\delta_S}(x) = \begin{cases} [1,1] & \text{if } x \in S \\ [0,0] & \text{if } x \notin S\,, \end{cases}$$

i.e.,

$$t_{\delta_S}(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases} \quad \text{and} \quad f_{\delta_S}(x) = \begin{cases} 0 & \text{if } x \in S \\ 1 & \text{if } x \notin S\,. \end{cases}.$$

Then δ_S *is called the vague characteristic set of* S *in [0, 1].*

Definition 2.8. *[2] A vague set* $A = (t_A, f_A)$ *of a* Γ*-semiring* R *is said to be vague* Γ*-semiring of* R *if it satisfies the following conditions: For all* $x, y \in R; \gamma \in \Gamma$*,*

(VI1) $V_A(x+y) \geq \min\{V_A(x), V_A(y)\}$;
(VI2) $V_A(x\gamma y) \geq \min\{V_A(x), V_A(y)\}$.

If A *is both left and right vague ideals of* R*, then* A *is called vague ideal of* R*.*

Definition 2.9. *[3] A vague set* $A = (t_A, f_A)$ *of a* Γ*-semiring* R *is said to be left (resp. right) vague ideal of* R *if it satisfies the following conditions: For all* $x, y \in R; \gamma \in \Gamma$*,*

(VI1) $V_A(x+y) \geq \min\{V_A(x), V_A(y)\}$;
(VI2) $V_A(x\gamma y) \geq V_A(y)$ *(resp.* $V_A(x\gamma y) \geq V_\psi(x)$*).*

If A is both left and right vague ideals of R, then A is called vague ideal of R.

Definition 2.10. *[4] A vague Γ-semiring $A = (t_A, f_A)$ of a Γ-semiring R is said to be vague bi-ideal if for all $x, y, z \in R; \alpha, \beta \in \Gamma$, $V_A(x\alpha y\beta z) \geq min\{V_A(x), V_A(z)\}$, i.e., $t_A(x\alpha y\beta z) \geq min\{t_A(x), t_A(z)\}$ and $f_A(x\alpha y\beta z) \leq max\{f_A(x), f_A(z)\}$.*

Definition 2.11. *[13] A vague Γ-semiring $\psi = (t_\psi, f_\psi)$ of R is said to be vague interior ideal of R if forall $x, y, z \in R; \alpha, \beta \in \Gamma$, $V_\psi(x\alpha y\beta z) \geq V_\psi(y)$ i.e., $t_\psi(x\alpha y\beta z) \geq t_\psi(y)$ and $1 - f_\psi(x\alpha y\beta z) \geq 1 - f_\psi(y)$.*

3. Vague Bi-interior Ideal of a Γ-semiring

In this section, we introduce and study vague bi-interior ideal as a generalization of vague bi-ideal, vague interior ideal of a Γ-semiring and characterize the vague bi-interior ideals of a Γ-semiring to a crisp bi-interior ideals of a Γ-semiring. Also, we prove that the intersection of vague bi-ideal and vague interior ideal of a Γ-semiring is a vague bi-interior ideal.

Throughout this section R stands for a Γ-semiring and δ stands for the vague characteristic set of R unless otherwise mentioned.

Now, we introduce the following.

Definition 3.1. *A vague Γ-semiring $A = (t_A, f_A)$ of R is called vague bi-interior ideal if $(\delta\Gamma A\Gamma\delta) \cap (A\Gamma\delta\Gamma A) \subseteq A$.*

Example 1. *Let R be the set of negative integers and Γ be the set of negative even integers. Then R, Γ are additive commutative semigroups.*
Define the mapping $R \times \Gamma \times R \to R$ by $x\alpha y$ usual product of x, α, y, $\forall\ x, y \in R; \alpha \in \Gamma$. Then R is a Γ-semiring.

Let $A = (t_A, f_A)$, where $t_A : R \to [0,1]$ and $f_A : R \to [0,1]$ defined by

$$t_A(x) = \begin{cases} 0.4 & \textit{if } \ x = -1 \\ 0.7 & \textit{if } \ x = -2 \\ 0.9 & \textit{if } \ x < -2 \end{cases} \quad \textit{and} \quad f_A(x) = \begin{cases} 0.4 & \textit{if } \ x = -1 \\ 0.2 & \textit{if } \ x = -2 \\ 0.1 & \textit{if } \ x < -2 \end{cases}.$$

Then A is a vague bi-interior ideal of R.

Theorem 3.1. *Every vague bi-ideal of R is a vague bi-interior ideal of R.*

Proof. Let $A = (t_A, f_A)$ be the vague bi-ideal of R. Then A is a vague Γ-semiring of R. Since A is vague bi-ideal, we have $A\Gamma\delta\Gamma A \subseteq A$. Now, $(\delta\Gamma A\Gamma\delta) \cap (A\Gamma\delta\Gamma A) \subseteq A\Gamma\delta\Gamma A \subseteq A$. Therefore $(\delta\Gamma A\Gamma\delta) \cap (A\Gamma\delta\Gamma A) \subseteq A$. Thus A is vague bi-interior ideal of R. □

Theorem 3.2. *Every vague interior ideal of R is a vague bi-interior ideal of R.*

Proof. Let $A = (t_A, f_A)$ be the vague interior ideal of R. Then A is a vague Γ-semiring of R. Since A is vague interior ideal, we have $\delta\Gamma A\Gamma\delta \subseteq A$. Now, $(\delta\Gamma A\Gamma\delta) \cap (A\Gamma\delta\Gamma A) \subseteq \delta\Gamma A\Gamma\delta \subseteq A$. Therefore $(\delta\Gamma A\Gamma\delta) \cap (A\Gamma\delta\Gamma A) \subseteq A$. Thus A is vague bi-interior ideal of R. □

Theorem 3.3. *Every left vague ideal of R is a vague bi-interior ideal of R.*

Proof. Let $A = (t_A, f_A)$ be the left vague ideal of R. Let $x \in R$. Now,

$$\begin{aligned} V_{\delta\Gamma A\Gamma\delta}(x) &= sup\{min\{V_{(\delta\Gamma A)}(p\alpha q), V_\delta(r)\}, where\ x = p\alpha q\beta r; \\ &\qquad p, q, r \in R\ and\ \alpha, \beta \in \Gamma\} \\ &= sup\{min\{V_{(\delta\Gamma A)}(p\alpha q)\}\} = sup\{min\{V_\delta(p), V_A(q)\}\} \\ &= sup\{V_A(q)\} \le sup\{V_A(p\alpha q)\} \\ &\le sup\{V_A(x)\} = V_A(x) \end{aligned}$$

That implies $\delta\Gamma A\Gamma\delta \subseteq A$. Also,

$$V_{A\Gamma\delta\Gamma A}(x) = sup\{min\{V_A(p), V_{\delta\Gamma A}(q\beta r)\},$$

where

$$x = p\alpha q\beta r; p, q, r \in R\ and\ \alpha, \beta \in \Gamma\} \le sup\{min\{V_A(p), V_A(q\beta r)\}\} = V_A(x).$$

That implies $A\Gamma\delta\Gamma A \subseteq A$. Therefore

$$V_{(\delta\Gamma A\Gamma\delta)\cap(A\Gamma\delta\Gamma A)}(x) = min\{V_{(\delta\Gamma A\Gamma\delta)}(x), V_{(A\Gamma\delta\Gamma A)}(x)\} \le V_A(x).$$

Thus A is a vague bi-interior ideal of R. □

Theorem 3.4. *Every right vague ideal of R is a vague bi-interior ideal of R.*

Proof. Proof is similar to the above theorem. □

Corollary 3.1. *Every vague ideal of R is a vague bi-interior ideal of R.*

Theorem 3.5. *A vague set $A = (t_A, f_A)$ is a vague bi-interior ideal of R if and only if its vague cut $A_{(\alpha,\beta)}$ is a bi-interior ideal of R.*

Proof. Suppose $A = (t_A, f_A)$ is a vague bi-interior ideal of R. Let $x \in (R\Gamma A_{(\alpha,\beta)}\Gamma R) \cap (A_{(\alpha,\beta)}\Gamma R\Gamma A_{(\alpha,\beta)})$. This implies $x \in R\Gamma A_{(\alpha,\beta)}\Gamma R$ and $x \in A_{(\alpha,\beta)}\Gamma R\Gamma A_{(\alpha,\beta)}$, i.e., $x = a\gamma p\eta b = q\zeta c\epsilon r$, where $a, b, c \in R;\ p, q, r \in A_{(\alpha,\beta)};\ \gamma, \eta, \zeta, \epsilon \in \Gamma$. Now,

$$\begin{aligned} V_{\delta\Gamma A\Gamma\delta}(x) &= sup\{min\{V_{\delta\Gamma A}(a\gamma p), V_\delta(b)\}\} \\ &= V_{\delta\Gamma A}(a\gamma p) = sup\{min\{V_\delta(a), V_A(p)\}\} \\ &= V_A(p) \geq [\alpha, \beta] \end{aligned}$$

Also,

$$\begin{aligned} V_{A\Gamma\delta\Gamma A}(x) &= sup\{min\{V_{A\Gamma\delta}(q\zeta c), V_A(r)\}\} \\ &= sup\{min\{sup\{min\{V_A(q), V_\delta(c)\}\}, V_A(r)\}\} \\ &\geq min\{V_A(q), V_A(r)\} \geq [\alpha, \beta]. \end{aligned}$$

Since A is vague bi-quasi ideal of R, we have $(\delta\Gamma A\Gamma\delta) \cap (A\Gamma\delta\Gamma A) \subseteq A$. This implies, $V_A(x) \geq V_{(\delta\Gamma A\Gamma\delta)\cap(A\Gamma\delta\Gamma A)}(x) = min\{V_{\delta\Gamma A\Gamma\delta}(x), V_{A\Gamma\delta\Gamma A}(x)\} \geq [\alpha, \beta]$, i.e., $x \in A_{(\alpha,\beta)}$.

Therefore $(R\Gamma A_{(\alpha,\beta)}\Gamma R) \cap (A_{(\alpha,\beta)}\Gamma R\Gamma A_{(\alpha,\beta)}) \subseteq A$. Hence $A_{(\alpha,\beta)}$ is bi-interior ideal of R.

Conversely, suppose that $A_{(\alpha,\beta)}$ is a bi-interior ideal of R. Obviously A is vague Γ-semiring of R. Suppose if possible $(\delta\Gamma A\Gamma\delta) \cap (A\Gamma\delta\Gamma A) \nsubseteq A$. That implies there exist $x \in R$ such that $V_A(x) < V_{(\delta\Gamma A\Gamma\delta)\cap(A\Gamma\delta\Gamma A)}(x)$. Let $[\alpha, \beta] \in [0, 1]$ such that

$$V_A(x) < [\alpha, \beta] < V_{(\delta\Gamma A\Gamma\delta)\cap(A\Gamma\delta\Gamma A)}(x). \tag{3.1}$$

Now, suppose for any $a, b, c, p, q, r \in R\ and\ \gamma, \zeta, \eta, \epsilon \in \Gamma$, $x = a\gamma p\zeta b = q\eta c\epsilon r$ such that $p, q, r \notin A_{(\alpha,\beta)}$. This implies that $V_A(p) < [\alpha, \beta],\ V_A(q) < [\alpha, \beta],\ V_A(r) < [\alpha, \beta]$. Now,

$$\begin{aligned} V_{(\delta\Gamma A\Gamma\delta)\cap(A\Gamma\delta\Gamma A)}(x) &= min\{V_{\delta\Gamma A\Gamma\delta}(x), V_{A\Gamma\delta\Gamma A}(x) \\ &= min\{sup\{min\{V_{\delta\Gamma A}(a\gamma p), V_\delta(b)\}\}, \\ &\qquad sup\{min\{V_{A\Gamma\delta}(q\eta c), V_A(r)\}\}\} \\ &= min\{V_{\delta\Gamma A}(a\gamma p), sup\{min\{V_{A\Gamma\delta}(q\eta c), V_A(r)\}\}\} \\ &= min\{sup\{min\{V_\delta(a), V_A(p)\}\}, sup\{min\{ \\ &\qquad sup\{min\{V_A(q), V_\delta(c)\}\}\}\}, V_A(r)\} \\ &= min\{V_A(p), V_A(q), V_A(r)\} \\ &< [\alpha, \beta] \end{aligned}$$

That implies $V_{(\delta\Gamma A\Gamma\delta)\cap(A\Gamma\delta\Gamma A)}(x) < [\alpha, \beta]$, which is a contradiction to (3.1). Therefore, there exist $a, b, c, p, q, r \in R$ *and* $\gamma, \zeta, \eta, \epsilon \in \Gamma$, $x = a\gamma p\zeta b = q\eta c\epsilon r$ such that $p, q, r \in A_{(\alpha,\beta)}$. Now, $a\gamma p\zeta b \in R\Gamma A_{(\alpha,\beta)}\Gamma R$ and $q\eta c\epsilon r \in A_{(\alpha,\beta)}\Gamma R\Gamma A_{(\alpha,\beta)}$. Hence, $x \in (R\Gamma A_{(\alpha,\beta)}\Gamma R) \cap (A_{(\alpha,\beta)}\Gamma R\Gamma A_{(\alpha,\beta)})$. This implie $x \in A_{(\alpha,\beta)}$. But $x \notin A_{(\alpha,\beta)}$, which is a contradiction to $A_{(\alpha,\beta)}$ is bi-interior ideal of R. Hence, $(\delta\Gamma A\Gamma\delta) \cap (A\Gamma\delta\Gamma A) \subseteq A$. Thus, A is a vague bi-interior ideal of R. □

Theorem 3.6. *Let B be a non-empty subset of R and $\delta_B = (t_{\delta_B}, f_{\delta_B})$ be the vague characteristic set of R. Then B is bi-interior ideal if and only if δ_B is vague bi-interior ideal of R.*

Proof. Suppose B is bi-interior ideal of R. Obviously δ_B is a vague Γ-semiring of R. Since B is bi-interior ideal, we have $(R\Gamma B\Gamma R) \cap (B\Gamma R\Gamma B) \subseteq B$. Now, $(\delta\Gamma\delta_B\delta) \cap (\delta_B\Gamma\delta\Gamma\delta_B) = \delta_{R\Gamma B\Gamma R} \cap \delta_{B\Gamma R\Gamma B} = \delta_{(R\Gamma B\Gamma B)\cap(B\Gamma R\Gamma B)} \subseteq \delta_B$. Thus, δ_B is a vague bi-interior ideal of R.

Conversely, suppose that δ_B is a vague bi-interior ideal of R. Then clearly $x + y \in B$, for all $x, y \in I$. Since δ_B is a vague bi-interior ideal of R, we have $(\delta\Gamma\delta_B\Gamma\delta) \cap (\delta_B\Gamma\delta\Gamma\delta_B) \subseteq \delta_B$. This implies $\delta_{R\Gamma B\Gamma B} \cap \delta_{B\Gamma R\Gamma B} \subseteq \delta_B$, i.e., $\delta_{(R\Gamma B\Gamma R)\cap(B\Gamma R\Gamma B)} \subseteq \delta_B$.

Therefore, $(R\Gamma B\Gamma R) \cap (B\Gamma R\Gamma B) \subseteq B$. Thus, B is a bi-interior ideal of R. □

Theorem 3.7. *If $A = (t_A, f_A)$ and $B = (t_B, f_B)$ are vague bi-interior ideals of R, then $A \cap B$ is a vague bi-interior ideal of R.*

Proof. Let A and B be a vague bi-interior ideals of R. Then $A \cap B$ is a vague Γ-semiring of R. Let $x \in R$. Now,

$$\begin{aligned}
V_{\delta\Gamma(A\cap B)}(x) &= sup\{min\{V_\delta(y), V_{A\cap B}(z), x = y\alpha z, where\ y, z \in R; \alpha \in \Gamma\}\} \\
&= sup\{min\{V_\delta(y), min\{V_A(z), V_B(z)\}\}\} \\
&= sup\{min\{min\{V_\delta(y), V_A(z)\}, min\{V_\delta(y), V_B(z)\}\} \\
&= min\{sup\{min\{V_\delta(y), V_A(z)\}\}, sup\{min\{V_\delta(y), V_B(z)\}\}\} \\
&= min\{V_{\delta\Gamma A}(x), V_{\delta\Gamma B}(x)\} = V_{(\delta\Gamma A)\cap(\delta\Gamma B)}(x)
\end{aligned}$$

That implies $\delta\Gamma(A \cap B) = (\delta\Gamma A) \cap (\delta\Gamma B)$. Also,

$$
\begin{aligned}
V_{(A\cap B)\Gamma\delta\Gamma(A\cap B)}(x) &= sup\{min\{V_{A\cap B}(y), V_{\delta\Gamma(A\cap B)}(z), x = y\alpha z, where\ y, z \in R; \alpha \in \Gamma\}\} \\
&= sup\{min\{V_{A\cap B}(y), V_{(\delta\Gamma A)\cap(\delta\Gamma B)}(z)\}\} \\
&= sup\{min\{min\{V_A(y), V_B(y)\}, min\{V_{\delta\Gamma A}(z), V_{\delta\Gamma B}(z)\}\}\} \\
&= sup\{min\{min\{V_A(y), V_{\delta\Gamma A}(z)\}, min\{V_B(y), V_{\delta\Gamma B}(z)\}\}\} \\
&= min\{sup\{min\{V_A(y), V_{\delta\Gamma A}(z)\}, sup\{min\{V_B(y), V_{\delta\Gamma B}(z)\}\}\} \\
&= min\{V_{A\Gamma\delta\Gamma A}(x), V_{B\Gamma\delta\Gamma B}(x)\} \\
&= V_{(A\Gamma\delta\Gamma A)\cap(B\Gamma\delta\Gamma B)}(x)
\end{aligned}
$$

That implies $(A \cap B)\delta\Gamma(A \cap B) = (A\Gamma\delta\Gamma A) \cap (B\Gamma\delta\Gamma B)$. Similarly we can prove $\delta\Gamma(A \cap B)\Gamma\delta = (\delta\Gamma A\Gamma\delta) \cap (\delta\Gamma B\Gamma\delta)$. Therefore

$$
\begin{aligned}
&[\delta\Gamma(A \cap B)\Gamma\delta] \cap [(A \cap B)\delta\Gamma(A \cap B)] \\
&\quad = (\delta\Gamma A\Gamma\delta) \cap (\delta\Gamma B\Gamma\delta) \cap (A\Gamma\delta\Gamma A) \cap (B\Gamma\delta\Gamma B) \subseteq A \cap B.
\end{aligned}
$$

Thus $A \cap B$ is a vague bi-interior ideal of R. □

Theorem 3.8. *The intersection of vague bi-ideal and vague interior ideal of R is a vague bi-interior ideal of R.*

Proof. Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be a vague bi-ideal and vague interior ideal of R respectively. Obviousely $A \cap B$ is a vague Γ-semiring of R. Now, $(A \cap B)\Gamma\delta\Gamma(A \cap B) \subseteq A\Gamma\delta\Gamma A \subseteq A$ and $\delta\Gamma(A \cap B)\Gamma\delta \subseteq B$. Therefore $[(A \cap B)\Gamma\delta\Gamma(A \cap B)] \cap [\delta\Gamma(A \cap B)\Gamma\delta] \subseteq A \cap B$. Thus $A \cap B$ is a vague bi-interior ideal of R. □

Theorem 3.9. *If $A = (t_A, f_A)$ is right vague ideal and $B = (t_B, f_B)$ is left vague ideal of R, then $A \cap B$ is a vague bi-interior ideal of R.*

Proof. Proof is clear from Theorem 3.3, 3.4 and 3.11. □

Theorem 3.10. *If $A = (t_A, f_A)$ is minimal left vague ideal and $B = (t_B, f_B)$ is minimal right vague ideal of R, then $C = A\Gamma B$ is a minimal vague bi-interior ideal of R.*

Proof. Suppose A is minimal left vague ideal and B is minimal right vague ideal of R. Let $x \in R$. Now,

$$
\begin{aligned}
V_{(\delta\Gamma C\Gamma\delta)\cap(C\Gamma\delta\Gamma C)}(x) &= min\{V_{\delta\Gamma C\Gamma\delta}(x), V_{C\Gamma\delta\Gamma C}(x)\} \leq V_{\delta\Gamma C\Gamma\delta}(x) \\
&= V_{(\delta\Gamma(A\Gamma B)\Gamma\delta)}(x) \leq V_{A\Gamma B}(x) = V_C(x).
\end{aligned}
$$

That implies $(\delta\Gamma C\Gamma\delta) \cap (C\Gamma\delta\Gamma C) \subseteq C$. Hence C is a vague bi-interior ideal of R.

Let G be a vague bi-interior ideal of R such that $G \subseteq C$. Now, $\delta\Gamma G \subseteq \delta\Gamma C = \delta\Gamma A\Gamma B \subseteq B$.

Similarly, we can prove $G\Gamma\delta \subseteq A$. Since A and B are minimal, we have $\delta\Gamma G = B$ and $G\Gamma\delta = A$. Also, $C = A\Gamma B = G\Gamma\delta\Gamma\delta\Gamma G \subseteq G\gamma\delta G$ and $C = A\Gamma B = A\Gamma\delta\Gamma G \subseteq \delta\Gamma G \subseteq \delta\Gamma G\Gamma\delta$. Therefore $C \subseteq (G\Gamma\delta\Gamma G) \cap (\delta\Gamma G\Gamma\delta) \subseteq G$. That implies $C = G$. Thus C is a minimal vague bi-interior ideal of R □

Theorem 3.11. *The intersection of vague bi-interior ideal and a vague Γ-semiring of R is also a vague bi-interior ideal of R.*

Proof. Let $A = (t_A, f_A)$ be a vague bi-interior ideal and $B = (t_B, f_B)$ be a vague Γ-semiring of R. Let $x \in R$. Now,

$$\begin{aligned}
V_{\delta\Gamma(A\cap B)\Gamma\delta}(x) &= sup\{min\{V_\delta(p), V_{(A\cap B)\Gamma\delta}(q\beta r),\\
&\qquad x = p\alpha q\beta r,\ where\ p, q, r \in R; \alpha, \beta \in \Gamma\}\}\\
&= sup\{min\{V_\delta(p), sup\{min\{V_{A\cap B}(q), V_\delta(r)\}\}\}\}\\
&= sup\{min\{V_\delta(p), sup\{min\{min\{V_A(q), V_B(q)\}, V_\delta(r)\}\}\}\}\\
&\leq sup\{min\{V_\delta(p), sup\{min\{V_A(q), V_\delta(r)\}\}\}\}\\
&= V_{\delta\Gamma A\Gamma\delta}(x)\,.
\end{aligned}$$

That implies $\delta\Gamma(A \cap B)\Gamma\delta \subseteq \delta\Gamma A\Gamma\delta$. Also,

$$\begin{aligned}
V_{(A\cap B)\Gamma\delta\Gamma(A\cap B)}(x) &= sup\{min\{V_{(A\cap B)}(p), sup\{min\{V_\delta(q), V_{A\cap B}(r)\}\},\\
&\qquad x = p\alpha q\beta r,\ where\ p, q, r \in R; \alpha, \beta \in \Gamma\}\}\\
&= sup\{min\{min\{V_A(p), V_B(p)\}, sup\{min\{V_\delta(q),\\
&\qquad min\{V_A(r), V_B(r)\}\}\}\}\}\\
&\leq sup\{min\{V_A(p), sup\{min\{V_{delta}(q), V_A(r)\}\}\}\}\\
&= V_{A\Gamma\delta\Gamma A}(x)\,.
\end{aligned}$$

That implies $(A\cap B)\Gamma\delta\Gamma(A\cap B)\subseteq A\Gamma\delta\Gamma A$. So, $[\delta\Gamma(A\cap B)\Gamma\delta]\cap[(A\cap B)\Gamma\delta\Gamma(A\cap B)]\subseteq(\delta\Gamma A\Gamma\delta)\cap(A\Gamma\delta\Gamma A)\subseteq A$. Moreover

$$\begin{aligned}V_{(A\cap B)\Gamma\delta\Gamma(A\cap B)}(x) &= sup\{min\{V_{(A\cap B)}(p), V_{A\cap B}(q), x = p\alpha q,\ where\ p, q\in R; \alpha\in\Gamma\}\}\\ &= sup\{min\{min\{V_A(p), V_B(p)\}, min\{V_A(q), V_B(q)\}\}\}\\ &\leq sup\{min\{V_B(p), V_B(q)\}\}\\ &\leq sup\{V_B(x)\}\\ &= V_B(x)\,.\end{aligned}$$

That implies $[\delta\Gamma(A\cap B)\Gamma\delta]\cap[(A\cap B)\Gamma\delta\Gamma(A\cap B)]\subseteq(A\cap B)\Gamma\delta\Gamma(A\cap B)\subseteq B$. Therefore $[\delta\Gamma(A\cap B)\Gamma\delta]\cap[(A\cap B)\Gamma\delta\Gamma(A\cap B)]\subseteq(A\cap B)\Gamma\delta\Gamma(A\cap B)\subseteq A\cap B$ Thus $A\cap B$ is a vague bi-interior ideal of R. □

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