

COMPARATIVE ANALYSIS OF TECHNIQUES OF ORDER REDUCTION FOR ANALYSIS OF VEHICLE MODEL

ANKUR GUPTA, AMIT KUMAR MANOCHA¹, AND GURPREET SINGH

ABSTRACT. It is essential to study the transfer function of the processes involved in any vehicle in order to know the behavioral study. As these are large operation processes and strenuous. In the light of this there is requirement to reduce the order of the vehicle's transfer function so that it becomes convenient and easy to analyze various behavioral parameters such as steady state error, settling time, peak overshoot etc. During the process of order reduction of these vehicle systems, it is desirable that the behavior of both original and reduced order system remains identical. So these constraints should be kept in mind by the researcher while designing and developing the model order reduction techniques to obtain the best possible approximation of the higher order system. This paper outlines hankel norm approximation, schur decomposition, normalized co-prime factor technique, balanced stochastic truncation techniques to reduce the order of a higher order system and then comparative study is undertaken for SISO and MIMO system by considering test examples on basis of performance parameters of time domain shown by step response behavior and frequency domain shown by bode plot. The comparative analysis of all the techniques is done to obtain the best technique out of the four techniques.

¹*corresponding author*

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1. INTRODUCTION

Every System has its transfer function and represented by a model. The same thing exists for a vehicle system and can be modeled by its transfer function. While developing the automotive industry, the main emphasis goes on the automobile quality and the same can be controlled by obtaining its model and then controlling the transfer function of that model [1]. developed the model of an automobile to reduce the longitudinal low frequency vibration of the order of 0-10Hz of that vehicle. The model includes torsional vibration of transmission system, tire, pitch and vertical vibration of the vehicle. In [2], the author extended the study for the suspension system of railway vehicle and obtained the control mechanism of it. The designed system replaced the passive secondary suspension employed in railway vehicles. The order of a model of suspension system of a vehicle is reduced by [3] from the 10 to 7 and 10 to 5, which also reduced the simulation time of system. The complex mechanical systems are converted accurately into the simple models in [4]. The complex model of a quarter car suspension system of 32nd order is reduced to its 4th order approximation by balanced realization technique, dominant pole technique and singular perturbation technique. From the study, it is observed that the transfer function of systems of vehicle processes is usually colossal and it is a laborious and challenging task to study the behavior of the system so model order is an essential part which enables the designers of the vehicle system to carry out study in an efficient and effective manner. The existing techniques of model order reduction have erroneous output between the original system and the reduced system which is undesirable [5]. This acts as the driving force to acquire a technique which would have least error with maximum performance output. The primary technique developed to perform the model order reduction was Balanced Truncation Technique [6] which formed the basis of all other techniques of model order reduction. To achieve higher accuracy of reduced system, various researches were performed and researchers developed different techniques to reduce the large scale system in the form of transfer function or state space. Hankel norm approximation technique [7] for order reduction of large scale systems uses the method for reduction of hankel error L_∞ among the higher order and reduced order system. A stochastic method [8] was developed on the basis of frequency weighted factor and [9] presented the projection

technique for the reduction of large scale MIMO systems. [10], [11] designed a novel approach known as schur decomposition to reduce the relative error among the original large scale model of state space form and reduced approximated system. Continued fraction expansion [12], PADE approximation [13], Stability equation method [14], Factor division [15], Response matching technique [16] techniques were developed for order reduction of integer order large scale LTI systems. Some real time problems were discussed by based on these techniques [17] - [19]. These techniques reduced the error among the original and reduced order system to a significant amount but possessed steady state error which was a major concern, so the need of development of some other technique was generated. Further, the developments were able to design the mixed approaches which combined two techniques together to form a mixed technique and achieved a better response with reduced amount of error and more accurate reduced order approximation of the large scale systems. In these techniques, numerator polynomial was reduced by one technique and denominator polynomial was reduced by another technique. [20] used the mixture of pole clustering technique and genetic algorithm to reduce the order of large scale MIMO system. An evolutionary computation based approach was used by [21], [22] as a mixture of Big Bang Big Crunch and dominant pole retention technique. Two techniques viz. Routh approximation and Cuckoo search algorithms were combined [23] to form the reduced order approximation by two previously designed techniques. The mixed techniques [20] - [28] designed were used to reduce the integral square error among the large scale and reduced scale system. These mixed approaches reduced the problem of earlier developed techniques as the reduced system obtained by mixed approaches possessed zero steady state error. But the mixed approaches were compared on the basis of integral square error only. Other factors were also required to be considered so that the best approximated reduced order system could be obtained from the large scale system. The Complete paper is divided into segments. The first section gives the basic introduction about the use of model order reduction for the vehicle models. Also this section highlights the previous literature in the field of model order reduction. The second section describes the methodology to perform the order reduction in the paper. The section describes all the techniques used to check the best possible ways to obtain the best reduction technique. After that, third section gives the results of the implementation of the studied

techniques on only SISO example and the results obtained are discussed in the same section. The last forth section outlines the conclusion of the study performed in the paper.

2. METHODOLOGY

Real time transfer function of a system is reduced to its second order approximated form using Hankel norm Approximation, Schur Decomposition, Normalized Co-prime factorization (NCF) and Balanced Stochastic Truncation (BST) Technique. Their effects are then compared on the basis of parameters namely peak overshoot, settling time and steady state error by applying step input to the system.

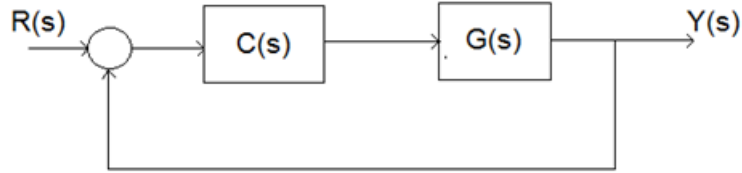


FIGURE 1. Block diagram for a closed loop system.

The first step execution to reduce the order of a higher order LTI system was to convert the given system into its state space representation [29] as shown in equation (2.1 & 2.2) and thus obtaining its state matrices (A, B, C, D).

$$(2.1) \quad \dot{x} = Ax + BU$$

$$(2.2) \quad y = Cx + DU$$

The reduction techniques are then applied to reduce the order of state matrices (A', B', C', D') and then these state matrices are converted back into the transfer function form using equation (2.3).

$$(2.3) \quad G_r(s) = C'(sI - A')^{-1} + D'$$

The paper includes four techniques which involves the reduction of A, B, C and D matrix for the reduction process and these four matrices are reduced to obtain A', B', C' and D' and then by putting the value in equation (2.3), reduced order transfer function is obtained. These techniques are Hankel norm Approximation, Schur Decomposition, Normalized Co-prime factorization (NCF) and

Balanced Stochastic Truncation (BST) Technique. These techniques are defined as under.

2.1. Hankel Norm Approximation. The problem of finding Hankel Norm Approximation [7] is to find the approximation $G_r(s)$ of McMillan degree $r < n$ so that the normalized error $\|G(s) - G_r(s)\|$ is minimized

$$(2.4) \quad P = \int_0^\infty e^{At} B B^T e^{A^T t} dt$$

$$(2.5) \quad Q = \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

The Technique of Hankel norm Approximation can be well understood as:

1. Find the controllability (P) and Observability (Q) grammian by using equation (2.4) and (2.5) after putting the value of required matrices in the equations.
2. Make sure that the controllability and observability grammian should satisfy the Lyapunov equations as given by equation (2.6) and (2.7).

$$(2.6) \quad P A^T + A P + B B^T = 0$$

$$(2.7) \quad Q A + A^T Q + C^T C = 0$$

3. By using matrix calculations, find the reduced order transfer matrices of A, B, C and D.
4. Using the transformed matrices, find the reduced order transfer function $G_r(s)$ using equation (2.3).

2.2. Balanced Stochastic Truncation (BST) Technique. Balanced stochastic Truncation [8] is the representation of the system with state covariance matrix which is equal to the canonical correlation coefficient matrix Σ for the output process. The following steps used to obtain the balanced stochastic realization:

1. Compute the matrices $V_{R,BIG}$ and $V_{L,BIG}$ whose columns form the basis for the respective right and left eigen spaces of P.Q associated with their 'big' eigenvalues.
2. Let

$$(2.8) \quad E_B I G = V_{L,BIG}^T \cdot V_{R,BIG}$$

and singular decomposition value is computed

$$(2.9) \quad U_{E,BIG} \sum_{E,BIG} V_{L,BIG}^T = E_B I G$$

3. The $S_{L,BIG}$ and $S_{R,BIG}$ matrices are calculated and hence the state space realization of Gr(s) is computed as

$$(2.10) \quad A_r = S_{L,BIG}^T \cdot A \cdot S_{R,BIG}$$

$$(2.11) \quad B_r = S_{L,BIG}^T \cdot B$$

$$(2.12) \quad C_r = C \cdot S_{R,BIG}$$

$$(2.13) \quad D_r = D$$

4. Using the equation (2.10)-(2.13) sequentially the reduced order approximation is found from equation (2.3)

2.3. Schur Decomposition Technique. This technique [11] makes use of controllability (P) and observability (Q) matrices for the reduction process but without using any balancing operation. This technique involves the process of dividing the matrix PQ in the right and left eigen spaces associated with big eigen values and then singular value decomposition of these two matrices is found out, using a set of equations, then the reduced order model state matrices are computed and hence the transfer function for the state model is designed. This technique can be better described by the following execution steps:

1. Firstly the matrix P.Q is converted into schur equivalent form by using VPQVT. Where V is an upper triangular matrix.
2. Then real transformation of V into ascending and descending order is computed from equation (2.14) and (2.15):

$$(2.14) \quad V_A^T \cdot PQ \cdot V_A = \begin{bmatrix} \lambda_1 & \bullet & \bullet & \bullet & \bullet \\ 0 & \lambda_2 & \bullet & \bullet & \bullet \\ 0 & 0 & \lambda_3 & \bullet & \bullet \\ M & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix}.$$

$$(2.15) \quad V_D^T . P Q . V_D = \begin{bmatrix} \lambda_n & \bullet & \bullet & \bullet & \bullet \\ 0 & \lambda_{n-1} & \bullet & \bullet & \bullet \\ 0 & 0 & \lambda_{n-2} & \bullet & \bullet \\ M & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \lambda_1 \end{bmatrix}.$$

3. Now V_A and V_D are portioned like as seen in equation (2.16) and (2.17).

$$(2.16) \quad V_A = \left\{ \begin{array}{c} n - k | k \\ V_{R(small)} | V_{L(big)} \end{array} \right\}$$

$$(2.17) \quad V_A = \left\{ \begin{array}{c} k | n - k \\ V_{R(big)} | V_{L(small)} \end{array} \right\}.$$

Then k big values ($V_{L(big)}$ and $V_{R(big)}$) are obtained from (2.16) and (2.17) are employed to find out the reduced system.

2.4. Normalized Co-prime Factor (NCF) Technique [17]. Consider the transfer function $G(s) = C(sE - A)^{-1}B$ of the system. Such transfer function can be represented in the form $G(s) = N(s).D(s)^{-1}$, where D is a square matrix and N is bearing the same matrix dimensions as $G(s)$. If, additionally, there exist X and Y such that $X(s).D(s) + Y(s).N(s) = I$, then D and N are called the right co-prime factors of G . If there exists a state feedback matrix F such that the pencil $(sE - A - BF)$ is stable and of index at most one, then the factors N and D can be chosen as

$$(2.18) \quad N(s) = C(sE - A - BF)^{-1}, D(s) = F(sE - A - BF)^{-1}B + I.$$

In this case the generalized state space representation of the extended transfer function $G_0(s) = [N(s)^T; D(s)^T - I]^T$ is given by

$$(2.19) \quad E\bar{x}(t) = (A + BF)x(t) + Bu_{ND}(t)$$

$$(2.20) \quad \begin{bmatrix} Y_N(t) \\ Y_D(t) \end{bmatrix} = \begin{bmatrix} C \\ F \end{bmatrix} x(t).$$

The original system can be represented as an combination of equation (2.19) and (2.20) which itself is subjected to the interconnection relation:

$$(2.21) \quad U_{ND}(t) = -y_{D(t)} + u(t) = \begin{bmatrix} 0, -1 \end{bmatrix} \begin{bmatrix} Y_N(t) \\ Y_D(t) \end{bmatrix} + u(t)$$

$$(2.22) \quad y(t) = y_{N(t)} = \begin{bmatrix} I, 0 \end{bmatrix} \begin{bmatrix} Y_N(t) \\ Y_D(t) \end{bmatrix}.$$

According to the formula of the transfer function for the closed loop system,

$$(2.23) \quad [I, 0](I - G - 0(s))[0, -I]^{-1}G_0(s) = N(s)D(s)^{-1} = G(s).$$

Thus the original system and the coupled system (2.19), (2.20) have the same transfer function. Hence the coupled system can be approximated by a reduced order system

$$(2.24) \quad W^T E T \bar{x}_r(t) = W^T (A + BF) T x_r(t) + W^T B u_{NDr}(t)$$

$$(2.25) \quad \begin{bmatrix} Y_{Nr}(t) \\ Y_{Dr}(t) \end{bmatrix} = \begin{bmatrix} CT \\ FT \end{bmatrix} x_r(t).$$

Here the projection matrices W, T are computed by the balanced truncation model reduction method and the transfer function is given by

$$G_{or}(s) = [N_r(s)^T, D_r(s)^T - I]^T$$

with

$$(2.26) \quad N_r(s) = CT(W^T(sE - A - BF)T)^{-1}W^T B + I$$

$$(2.27) \quad D_r(s) = FT(W^T(sE - A - BF)T)^{-1}W^T B + I$$

and the norm error bound is

$$(2.28) \quad \|G_{or}(s) - G_0(s)\| = \left\| \begin{bmatrix} N_r & -N \\ D_r & -D \end{bmatrix} \right\|.$$

So $G_{or}(s)$ system gives the reduced order approximation of original $G_0(s)$ system.

3. RESULTS AND DISCUSSION

Firstly, an eighth (8^{th}) order system with the transfer function [13], [24] given by equation (3.1) is selected for carrying out the comparative analysis:

$$(3.1) \quad T = \frac{18s^7 + 514s^6 + 5982s^5 + 36380s^4 + 122664s^3 + 185760s^2 + 185760s + 40320}{s^8 + 36s^7 + 546s^6 + 4536s^5 + 22449s^4 + 67284s^3 + 118124s^2 + 1095840s + 40320}.$$

The system represented by equation (3.1) is reduced to get the 2nd order reduced approximation using all the four techniques mentioned and then the transfer function obtained after reduction is given in Table 1.

TABLE 1. Transfer function obtained after applying reduction techniques.

Sr. No.	Technique used	Reduced order transfer function
1	Hankel Norm Approximation	$\frac{15.73s+4.649}{s^2+6.456s+5.349}$
2	Schur Decomposition	$\frac{17.92s+5.342}{s^2+7.534s+5.501}$
3	NCF Technique	$\frac{17.6s+4.908}{s^2+7.206s+5.308}$
4	BST technique	$\frac{18s+5.141}{s^2+7.528s+5.369}$

To compare the original and reduced system, the results are simulated for both the systems using MATLAB. Bode plot of the original and reduced system is as shown in Figure 2. Bode plot behavior in magnitude as well as phase analysis shows that the frequency domain response of the original system and reduced systems obtained by all the four techniques is almost same with not any significant error. The Gain Margin and Phase Margin for both the original and reduced system are approximately same. This shows that frequency response of the system don't change much after reducing its transfer function order from 8th order to even 2nd order. Hence, these reduced approximations can be used to replace the original system with its reduced approximation.

Further the both type of systems (original and reduced) are compared through their time domain behavior with the step input applied at the input port of the system. Time domain behavior using the MATLAB is as shown in figure 3. The study of time domain behavior for step input from figure 3 shows that many parameters of reduced and original systems are same. Peak overshoot is almost same in all the cases but the settling time and steady state error of the reduced system has been changed for all the four techniques. The amount of peak overshoot, settling time and steady state error occurring in the reduced

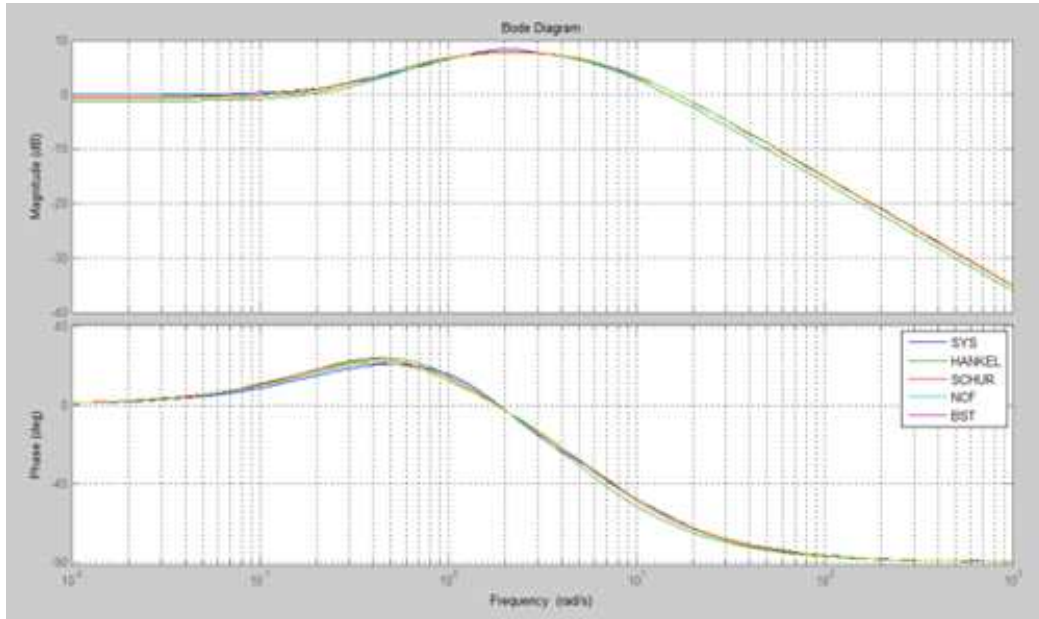


FIGURE 2. Bode Plot of original and reduced system obtained by all techniques.

system obtained by all the four techniques are as shown in Table 2, which shows that peak overshoot is almost same in all the four techniques but steady state error is least in NCF technique of model order reduction with settling time little higher than Hankel Norm approximation. The BST technique has higher settling time than all the four techniques which is not desirable. Hankel Norm technique has the minimum amount of settling time but highest steady state error which makes it less useful.

TABLE 2. Value of peak overshoot, settling time and steady state error for all techniques.

Sr. No.	Technique used	Peak Overshoot	Settling time (seconds)	Steady state error
1	Hankel Norm Approximation	53.70	6.71	0.127
2	Schur Decomposition	54.73	7.79	0.072
3	NCF Technique	53.91	7.53	0.025
4	BST technique	54.12	9.68	0.042

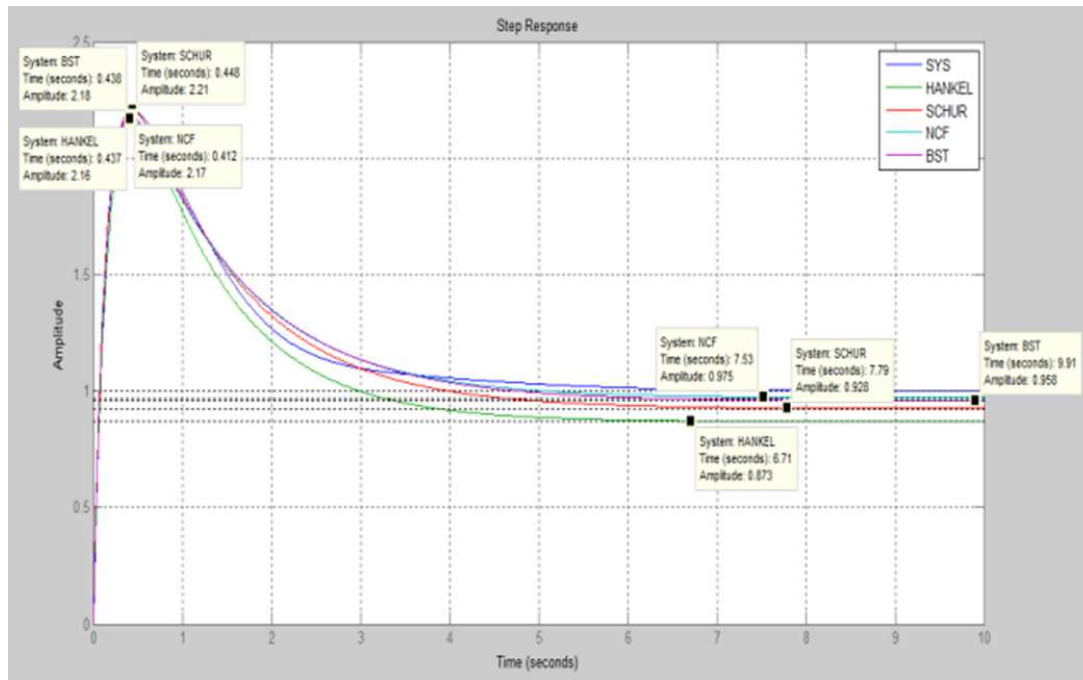


FIGURE 3. Transient response of original and reduced systems obtained by all techniques.

It is necessary to study the behavior of the vehicle system before it is implemented in any automotive industry. Also the transfer function of these vehicle systems is large and required to reduce for proper behavioral analysis. The study of all the four techniques mentioned in the paper reveals that, it is clear that behavior of reduced order system is more convenient to study. Also, the study of the bode plot analysis shows that the reduction of higher order system makes less effect on the frequency domain characteristics of the original system, but on the other side while viewing the transient response behavior of the original and reduced system, it is clear that some of the parameters gets changed and hence generating the error between the original and reduced system. This error must be reduced in order to get the exact approximation of the original system which is to be reduced. So, either by less reduction in the order or by using the appropriate technique among the four techniques mentioned in the paper, the most suitable reduced order approximation of the vehicle system can be obtained.

4. CONCLUSION

The techniques of model order reduction should be applied to the vehicle system as each vehicle has a particular transfer function and the knowledge of behavior of that transfer function is very much required. This behavioral study can be made possible by order reduction of that vehicular system model. So the study can be further extended by applying the best techniques obtained on a particular vehicle model to study its behavior and make changes accordingly to make the vehicular system more convenient and making transportation easy. Also the current study can be further implemented to develop a new technique in the future which has the least impact on the system indices with upgraded performance parameters.

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DEPARTMENT OF ELECTRONICS AND COMMUNICATION ENGINEERING
MAHARAJA RANJIT SINGH PUNJAB TECHNICAL UNIVERSITY
BATHINDA, PUNJAB, INDIA
Email address: ankurgarg2711@gmail.com

DEPARTMENT OF ELECTRICAL ENGINEERING,
PUNJAB INSTITUTE OF TECHNOLOGY
GTB GARH, MOGA, PUNJAB, INDIA
Email address: manocha82@gmail.com

DEPARTMENT OF COMPUTER SCIENCE,
PUNJAB INSTITUTE OF TECHNOLOGY
RAJPURA, PUNJAB, INDIA
Email address: gps_ynr@yahoo.com