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RIEMANN INTEGRAL vs. LEBESGUE INTEGRAL: A PERSPECTIVE VIEW

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ABSTRACT. Integration theory is one of the most widely employed mathematical techniques in diverse disciplines that has served as foundation of Analysis. Many researchers have participated and contributed to the progressive journey of the development of integration theory by making improvements in the existing theories to establish a generalized theory following existing norms and taking into account earlier limitations. Bernhard Riemann's theory, commonly know as Riemann Integral and Henri Lebesgue's theory, known as Lebesgue Integral are the most commonly used approaches for integration and are considered as building blocks for the current integration theory. This article aims at discussing the Riemann and Lebesgue approaches for integration of a function along with its limitations. Major contrasts between the two integration techniques are also highlighted with suitable examples. The paper will contribute to future research on similar topics. The research reflects that although both the Riemann integral and the Lebesgue integral are strong integration techniques, still there is a scope of research in finding better integration theory that can provide better versions of many associated theorems including the Fundamental Theorem of Calculus, which is discussed in detail in this paper $w.r.t.$ both the integrals.

1. INTRODUCTION

Integration is the fundamental concept of Analysis which is widely acceptable in pure and applied mathematics. The findings in Analysis in the early fragment of 20th century made this span distinctive as it is credited for the origination

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of the present-time theory of functions which sneak deep into several branches of mathematics [1]. It can be utilized for finding quantities like area, mass, volume, work, moments of inertia etc. Not only various differential and integral equations are solved using it but also it is used for distribution in Statistics [2]. In fact, modern integration is the outcome of filtration and modification of ideas of earliest Egyptian and Greek mathematicians.

2. History and Development

Greeks are credited for the origination of the process of Integration. Eudoxus, the Greek astronomer used the geometry of figures to fit a sequence of non-overlapping triangles inside the given region to cover the whole area of the region. Areas of circles and similar figures were found by Archimedes after about a century, using the concept of limits. Later in 17th century, Descartes invented graphs. At about the same time, Newton and Leibnitz invented Calculus [3]. Riemann first published about definite integral on the real line in 1868. Riemann's method could not be operated on integrals of sines and cosines for bounded functions and even on class of highly discontinuous functions. These limitations of Riemann integral served as foundation of Lebesgue integral. Lebesgue's work was highly inspired by Fourier series and since then Lebesgue integral began to flourish.

3. Riemann Integration

Bernhard Riemann defined integral in terms of a limit of Riemann sums. Let $[a, b] \subset \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ be the function whose integration is to be performed. Subdivisions of $[a, b]$ are formed by non-overlapping subintervals $[x_{i-1}, x_i]$ $s.t.$ $a = x_0 < x_1 < ...... < x_{n-1} < x_n = b$. Let $P = \{x_0, x_1, x_2, ..., x_n\}$ denote partition of $[a, b]$. Choose a real number w_i in each $[x_{i-1}, x_i]$ for $i = 1, 2, ..., n$. The sum $\sum_{i=1}^{n} f(w_i)(x_{i-1}, x_i)$ which is defined $w.r.t.$ P and the set of sampling points w_i, $1 \leq i \leq n$ is called Riemann sum of f on $[a, b]$. These chosen Riemann sums congregate around a number called the Riemann integral. Mathematically, $\mathcal{R}$ is $s.t.b.$ Riemann intergal of f on $[a, b]$, if to every positive number ϵ, $\exists$ a positive number δ $s.t.$ whenever $\max\limits_{1 \leq i \leq n} |x_i - x_{i-1}| < \delta \Rightarrow |\mathcal{R} - \sum_{i=1}^{n} f(w_i)(x_{i-1}, x_i)| < \epsilon$.

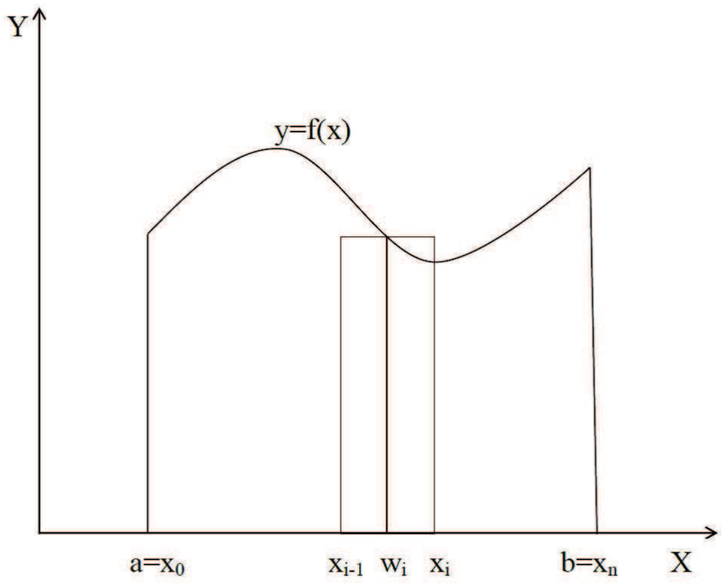

FIGURE 1. Graphical representation of function $y = f(x)$ under Riemann Integration.

Figure 1 depicts only one rectangle made using sampling point w_i in the $i^{th}-$subinterval. When all such rectangles are drawn and added, the Riemann sum is formed. As $n \to \infty$, the limiting value, if it exists is given by

$$\mathcal{R} = \int_a^b f(x)dx = \lim_{\max\limits_{1\le i\le n} |x_i - x_{i-1}| \to 0} \sum f(wi)\Delta x_i$$

3.1. **Limitations.** The main shortcoming of Riemann integration is that it requires too much regularity of the function which in turn restricts most of the convergence theorems for Riemann integral. Its other limitation is that this integral is applicable only on bounded functions and bounded intervals [4]. The other limitations are mentioned in the section of Comparison between Lebesgue and Riemann integral.

4. LEBESGUE INTEGRAL

Lebesgue integration, the powerful tool used in advancements in real analysis, modern function theory, probability theory and statistics was introduced by

Henri Lebesgue, a French mathematician in 1904. He used strong and conceptual base of 'Measure theory' that aids in the clear perception of sizes of sets and then builds integral for a simple function that partitions the range into sets. These integrals of simple functions are then employed to determine integral for a general class of functions $\mathcal{L}'$ [5].

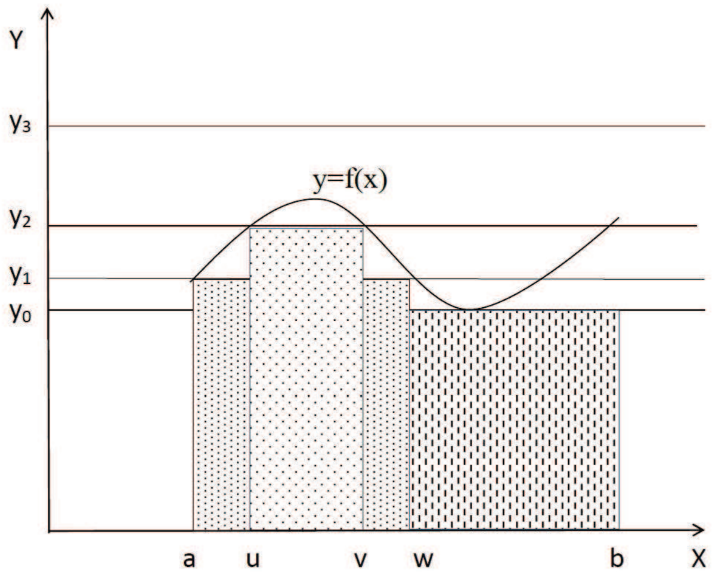

FIGURE 2. Graphical representation of function $y = f(x)$ under Lebesgue Integration

In Figure 2, the area is given by $\sum_{i=0}^{2} y_i\, m(A_{i+1})$, where
$A_1 = \{x \in [a, b]; y_0 \le f < y_1\}$ = the interval wb
$A_2 = \{x \in [a, b]; y_1 \le f < y_2\}$ = union of disjoint intervals au and vw
$A_3 = \{x \in [a, b]; y_2 \le f < y_3\}$ = the interval uv
and m represents the measure of the set [6].

Here the range of function is subdivided instead of subdividing that interval on which the function is defined. If $f : \mathbb{R}^n \to \mathbb{R}$ be a non-negative simple function whose canonical representation is $f = \sum_{i=1}^{m} b_i \chi_{E_i}$, where the sets Ei are measurable and pairwise disjoint then Lebesgue interval of f is given by $\int_{\mathbb{R}^n} f = \sum_{i=1}^{m} b_i\, m(E_i)$. Starting from simple functions, this definition is further extended to non-negative measurable functions.

4.1. **Limitations.** The definition of integral given by Lebesgue is not suitable enough for the introduction purposes. Lebesgue integration requires the tedious study of 'Measure theory' and 'Sigma algebra' before it defines integral. There are some functions for which improper Riemann integral exists but Lebesgue integral doesn't. One such function is $\frac{\sin x}{x}$ on $[0, \infty)$.

5. Comparison between Riemann and Lebesgue Integral

The difference between both the approaches can be summarized as:- Let there be a certain amount of money that has been accumulated. Out of it, some fixed amount is to be paid to the creditor. Riemann's approach is to take the rupees and paise in the same order as they are kept until that sum of fixed amount is reached. But Lebesgue's approach is to order rupees and paise according to identical values, pile them up accordingly and then to add all these piles and to pay the required sum to the creditor [7].

The Fundamental theorem of Calculus states that 'If a continuous function F is differentiable at each point on $[a,b]$, and if its derivative F' is Riemann integrable on $[a,b]$ then

$$\int_a^x F'(x) = F(x) - F(a) \ ,\forall x \in [a,b]."$$

It is very obvious from the above statement that all the derivatives (bounded or unbounded) may not be Riemann integrable. But we know that all the bounded derivatives are Lebesgue integrable. So the Fundamental theorem of Calculus may be restated as 'If F is differentiable on $[a,b]$ and if its derivative F' is bounded on $[a,b]$, then F' is Lebesgue integrable on $[a,b]$ and $\int_a^x F'(x) = F(x) - F(a), \forall x \in [a,b]$.' This better version of Fundamental theorem of Calculus is due to theory of Lebesgue integration [8].

Lebesgue's work is distinguished by its productive use of term-by-term integration of sequence and series of functions that are not convergent uniformly. But in Riemann integration, a rigid restriction like that of uniform convergence is required so that the limit of integrals approaches the integral of limit. In the functions that are Lebesgue integrable, exchange of limit and integral

is possible but Riemann integrable functions lack this abstract feature of exchange. Bounded convergence theorem (BCT) and Monotone convergence theorem (MCT) supports this exchange in Lebesgue integration but not always for Riemann integration. Example- Let $\{ r_i \}$ be a sequence of all rational numbers in $[0, 1]$. Let $t_n = \{ r_i; i = 1, 2, 3, ..., n \}, n \in \mathbb{N}$
Define for each $n \in \mathbb{N}$, the function $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1, & if\, x \in t_n \\ 0, & if\, x \notin t_n \end{cases}$$

$\because f_n$ has only n points of discontinuities, $\therefore f_n$ is $\mathcal{R}$-integrable on $[a, b]$, $\forall\, n \in \mathbb{N}$.
Further $f_n \to f$ on $[0, 1]$ where limit function f is given by

$$f(x) = \begin{cases} 1, & if\, x \in \mathbb{Q} \\ 0, & if\, x \in \mathbb{R} - \mathbb{Q}, \end{cases}$$

which is not Riemann integrable.

Similarly, Dominated convergence theorem (DCT) can't be applied to Riemann-integrable functions. Although there are versions of BCT and MCT for Riemann integral but in that case limit function has to be Riemann integrable. The beauty of DCT is that it can be applied to any sequence which is bounded above by integrable function .

These convergence theorems put Lebesgue's theory of integration in a superior position. Also, these convergence theorems including Fatou's Lemma stay valid under 'Convergence in Measure', hence paving the way for further extension of Lebesgue integration [9].

The result 'A measureable function f is integrable over E iff $|f|$ is integrable over E.' holds in case of Lebesgue-integration but fails to hold in case of Riemann integration. Let $f : [0, 1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1, & if\, x \in \mathbb{Q} \\ -1, & if\, x \in \mathbb{R} - \mathbb{Q}, \end{cases}$$

Now Now $|f| = 1$ being constant, is integrable. But $f(x)$ is not $\mathcal{R}$-integrable.

Existence of functions that are not Riemann integrable but are Lebesgue integrable:-

- The characteristic function of rational numbers in $[0,1]$ $i.e.$, the Dirichlet function given by

$$f(x)=\begin{cases}1, & if\, x\in\mathbb{Q}\\ 0, & otherwise,\end{cases}$$

is not Riemann integrable but is Lebesgue integrable with value as zero. In Lebesgue's theory, one needs to find out here the amount of space of the interval that is equal to 1 and the amount of space that is equal to zero.
- Another example is $f(x)=\frac{2}{\sqrt{x}}$ over $[0,1]$. Being unbounded, it is not Riemann integrable but still is Lebesgue integrable.

Also in terms of extension, extension of Lebesgue integral is easy to understand but extension of Riemann integral to multiple integrals is unmanageable.

6. Conclusion and Future Scope

It is concluded that the Lebesgue integral reinstates the logic and rationality of the 'Fundamental Theorem of Calculus'. As discussed above, Riemann integral doesn't support the powerful standard Convergence theorems like BCT, MCT and others. Therefore, the useful processes of Real Analysis like limits do not fit in well within the realm of Riemann integration. Lebesgue integration is more flexible and is actually an extension of Riemann integration in the true sense. A complete space can be constructed by using equivalence classes of $\mathcal{L}$-integrable functions, making it superior than $\mathcal{R}$-integrals. But still the work of all those who worked before Lebesgue should not be dishonoured. The work done before Lebesgue had ingenuity and uniqueness which was unparallel.

Lastly, consider

$$f(x)=\begin{cases}x^2\cos\frac{\pi}{x^2} & ;\, 0<x\leq 1\\ 0 & ;\, x=0,\end{cases}$$

which is differentiable but its derivative is not $\mathcal{L}$-integrable. Also, its derivative, being unbounded is not $\mathcal{R}$-integrable. Hence, Fundamental Theorem of Calculus- I does not go well with both Riemann integral and Lebesgue integral.

To seek better and more generalized versions of Fundamental Theorem of Calculus and other related theorems, more research can be carried out by taking into account the comparisons in new evolving integration techniques and by deep learning of integrals like Henstock-Kurzweil integral and others.

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