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OPTIMAL ESTIMATION FOR THE NORM OF PRE-SCHWARZIAN DERIVATIVE

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ABSTRACT. In this research contribution we have considered two subclasses of bi-univalent functions defined using subordination and studied about the bounds for the pre-Schwarzian norm. Initially Shalini et al. have handled this problem. We have made a remark on the proofs and bounds by Shalini et al.

1. INTRODUCTION

Let H be the class consisting of all analytic functions on the unit disk U where $U = \{z : z \in \mathbb{C}, |z| < 1\}$. Consider the subclass A of H, that comprises of all analytic functions that are normalized in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let

$$L_u = \{f \in A : f'(z) \neq 0, z \in U\},$$

the class of all locally univalent functions. The norm $\| T_f \|$ of the pre-Schwarzian derivative of $f \in L_u$ is defined by

$$\| T_f \| = \sup_{|z|<1} \left(1 - |z|^2\right) \left| \frac{f''(z)}{f'(z)} \right|.$$

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It is essential to note that $\| T_f \| < \infty$ iff f is uniformly locally univalent in U. It is also to be noted that if $f \in S$, the class of univalent functions, then $\| T_f \| \leq 6$. Conversely it follows from Becker's theorem, that if $f \in A$ and if $\| T_f \| \leq 1$, then $f \in S$. The results are sharp [2, 3]. For functions belonging to the class of Convex functions C, $\| T_f \| \leq 4$. According to Yamashita [12], if $f \in S^*(\alpha)$, then $\| T_f \| \leq 6 - 4\alpha$. Bhowmik et. al. [4] have obtained an estimate of the norm to be $4 \leq \| T_f \| \leq 2\alpha + 2$ for functions in the class of Concave univalent functions of order α.

The norm has a vital role to play with the theory of Teichmüller spaces. It is considered to be an element of Complex Banach Space in the theory of Teichmüller spaces.

For $f_1(z)$ and $f_2(z) \in A$, if there is a Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ in U such that $f_1(z) = f_2(w(z))$, we say that $f_1(z)$ is subordinate to $f_2(z)$ and we write $f_1(z) \prec f_2(z)$.

A function $f(z)$ in S belongs to σ, the class of all bi-univalent functions, if both $f(z)$ and its inverse has an analytic continuation to $|w| < 1$.

Lewin [7] introduced the class of bi-univalent functions in 1967 and gave an estimate for the second coefficient for functions belonging to this class as $|a_2| < 1.51$. His result was improved by Brannan and Clunie [5] to $|a_2| \leq \sqrt{2}$. There exists a substantial literature on the estimation of the initial coefficients of bi-univalent functions, see [1, 11, 13, 14].

Following are the theorems proved by Shalini et. al [10] [Theorem 2.1 and Theorem 2.2]. We have made a remark on these two theorems and presented the modified version of the same.

Theorem 1.1. *Let f represented by* (1.1) *be $\mathcal{S}^*{}_\sigma[\mathfrak{A}, \mathfrak{B}]$, then*

$$\| T_f \| \leq min \left(\frac{2(\mathfrak{A} - \mathfrak{B})(\mathfrak{A} + 2)}{\mathfrak{A} + 1}, \frac{2(\mathfrak{A} - \mathfrak{B})|\mathfrak{A}|}{\mathfrak{A} + 1} \right).$$

Theorem 1.2. *Let f represented by* (1.1) *be in $\mathfrak{V}^*{}_\sigma[\mathfrak{A}, \mathfrak{B}]$, then*

$$\| T_f \| \leq min \left(\frac{2(3 + 2\mathfrak{A})(\mathfrak{A} - \mathfrak{B})}{\mathfrak{A} + 1}, \frac{2(\mathfrak{A} - \mathfrak{B})(1 + 2\mathfrak{A})}{\mathfrak{A} + 1} \right).$$

2. Definitions

Definition 2.1. *Let f be represented by* (1.1). *Then f is in the class ${\mathcal{S}^*}_\sigma[\mathfrak{A},\mathfrak{B}]$, if*

$$\frac{zf'(z)}{f(z)} \prec \frac{1+\mathfrak{A}z}{1+\mathfrak{B}z} \tag{2.1}$$

and

$$\frac{wg'(w)}{g(w)} \prec \frac{1+\mathfrak{A}w}{1+\mathfrak{B}w},$$

where $f \in \sigma$, $g = f^{-1}$, $w = f(z)$, $w \in U$ and $-1 \le \mathfrak{B} < \mathfrak{A} \le 1$.

Definition 2.2. *A function f of the form* (1.1) *is in ${\mathfrak{V}^*}_\sigma[\mathfrak{A},\mathfrak{B}]$, if*

$$\frac{z^2 f'(z)}{[f(z)]^2} \prec \frac{1+\mathfrak{A}z}{1+\mathfrak{B}z}, \tag{2.2}$$

and

$$\frac{w^2 g'(w)}{[g(w)]^2} \prec \frac{1+\mathfrak{A}w}{1+\mathfrak{B}w}, \tag{2.3}$$

where $f \in \sigma$, $g = f^{-1}$, $w = f(z)$, $w \in U$ and $-1 \le \mathfrak{B} < \mathfrak{A} \le 1$.

3. Main Results

3.1. Norm estimation for the subclass ${\mathcal{S}^*}_\sigma[\mathfrak{A},\mathfrak{B}]$.

Theorem 3.1. *Let f be given by* (1.1). *If f is in the class $\mathcal{S}^*_\sigma[\mathfrak{A},\mathfrak{B}]$, then*

$$\| T_f \| \le \begin{cases} \frac{2(\mathfrak{A}-\mathfrak{B})}{1+\mathfrak{A}}, & -1 < \mathfrak{A} < 0, \\ 2|\mathfrak{B}|, & \mathfrak{A} = 0, \\ \min\left(\frac{2(\mathfrak{A}-\mathfrak{B})(\mathfrak{A}+2)}{1+\mathfrak{A}}, \frac{2(\mathfrak{A}-\mathfrak{B})}{1-\mathfrak{A}}\right), & 0 < \mathfrak{A} < 1, \\ 3(1-\mathfrak{B}), & \mathfrak{A} = 1. \end{cases}$$

Proof. Since $f \in \mathcal{S}^*_\sigma[\mathfrak{A},\mathfrak{B}]$, the Mobius transformation $\dfrac{1+\mathfrak{A}z}{1+\mathfrak{B}z}$, $-1 \le \mathfrak{B} < \mathfrak{A} \le 1$ transforms the unit disc onto another disc (or half plane) with end points of the diameter as $\left(\dfrac{1-\mathfrak{A}}{1-\mathfrak{B}}, \dfrac{1+\mathfrak{A}}{1+\mathfrak{B}}\right)$. As (2.1) holds, by [10] we

have taken $\| T_f \| \leq \dfrac{2(\mathfrak{A}-\mathfrak{B})(\mathfrak{A}+2)}{\mathfrak{A}+1}$.
On the other hand,

$$\frac{wg'(w)}{g(w)} \prec \frac{1+\mathfrak{A}w}{1+\mathfrak{B}w} = s(w), \tag{3.1}$$

where $z = f^{-1}(w) = g(w)$. A Schwarz function $\varphi : U \to U$ exists by the definition of subordination with $\varphi(0) = 0$ and $|\varphi(z)| < 1$, such that

$$v(z) = s \circ \varphi(z) = \frac{1+\mathfrak{A}\varphi(z)}{1+\mathfrak{B}\varphi(z)}.$$

Since $f \in \sigma$, both f and its inverse are analytic and univalent in U. The derivative of f^{-1} (see pp. 1038 [9]) is given by

$$\frac{d(f^{-1}(w))}{dw} = \frac{1}{f'(z)}.$$

Therefore (3.1) can be rewritten as

$$\frac{f(z)}{zf'(z)} = \frac{1+\mathfrak{A}\varphi(z)}{1+\mathfrak{B}\varphi(z)},$$

or equivalently

$$\frac{f'(z)}{f(z)} = \frac{1+\mathfrak{B}\varphi(z)}{z(1+\mathfrak{A}\varphi(z))}, \quad (z \in U). \tag{3.2}$$

The logarithmic differentiation of (3.2), gives

$$\frac{f''(z)}{f'(z)} = \frac{(\mathfrak{B}-\mathfrak{A})\varphi(z)}{z(1+\mathfrak{A}\varphi(z))} + \left(\frac{\mathfrak{B}}{1+\mathfrak{B}\varphi(z)} - \frac{\mathfrak{A}}{1+\mathfrak{A}\varphi(z)}\right)\varphi'(z). \tag{3.3}$$

According to Schwarz-Pick lemma, a Schwarz function $\varphi(z)$ can be obtained such that

$$|\varphi'(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2} \quad (z \in U). \tag{3.4}$$

By (3.4) and as $|\varphi(z)| \leq |z|$ in [6], the relation (3.3) implies that

$$\begin{aligned}
\left|\frac{f''(z)}{f'(z)}\right| &= \left|\frac{(\mathfrak{B}-\mathfrak{A})\varphi(z)}{z(1+\mathfrak{A}\varphi(z))} + \frac{(\mathfrak{B}-\mathfrak{A})\varphi'(z)}{(1+\mathfrak{A}\varphi(z))(1+\mathfrak{B}\varphi(z))}\right|, \\
&\leq \frac{|\mathfrak{B}-\mathfrak{A}||\varphi(z)|}{|z|(1-|\mathfrak{A}||\varphi(z)|)} + \frac{|\mathfrak{B}-\mathfrak{A}||\varphi'(z)|}{(1-|\mathfrak{A}||\varphi(z)|)(1-|\mathfrak{B}||\varphi(z)|)}, \\
&\leq \frac{|\mathfrak{B}-\mathfrak{A}|}{(1-|\mathfrak{A}||z|)} + \frac{|\mathfrak{B}-\mathfrak{A}|(1+|z|)}{(1-|\mathfrak{A}||z|)(1-|z|^2)}.
\end{aligned}$$

We have

$$\begin{aligned}\| T_f \| &= \sup_{|z|<1} \left(1-|z|^2\right) \left|\frac{f''(z)}{f'(z)}\right|, \\ &\leq \sup_{|z|<1} \left(\frac{|\mathfrak{B}-\mathfrak{A}|\left((1-|z|^2)+(1+|z|)\right)}{(1-|\mathfrak{A}||z|)}\right), \\ &= \frac{2|\mathfrak{B}-\mathfrak{A}|}{1-|\mathfrak{A}|} =: \varphi(\mathfrak{A},\mathfrak{B}).\end{aligned}$$

Here are the bounds for the norm, for various choices of $\mathfrak{A}$.

Case 1. Let $\mathfrak{A}=0$. Then we have $\varphi(\mathfrak{A},\mathfrak{B}) = 2|\mathfrak{B}|$ and

$$\| T_f \| \leq \min\left(-4\mathfrak{B}, 2|\mathfrak{B}|\right) = 2|\mathfrak{B}|, \quad (-1 \leq \mathfrak{B} < 0).$$

Case 2. When $\mathfrak{A}=1$ we have $\varphi(\mathfrak{A},\mathfrak{B}) = \infty$. So

$$\| T_f \| \leq \min\left(3(1-\mathfrak{B}), \infty\right) = 3(1-\mathfrak{B}).$$

Case 3. If $0<\mathfrak{A}<1$, then $\varphi(\mathfrak{A},\mathfrak{B})$ becomes $\frac{2(\mathfrak{A}-\mathfrak{B})}{1-\mathfrak{A}}$. We have

$$\| T_f \| \leq \min\left(\frac{2(\mathfrak{A}-\mathfrak{B})(\mathfrak{A}+2)}{1+\mathfrak{A}}, \frac{2(\mathfrak{A}-\mathfrak{B})}{1-\mathfrak{A}}\right).$$

Case 4. When $\mathfrak{A}$ lies between -1 and 0, $\varphi(\mathfrak{A},\mathfrak{B}) = \frac{2|\mathfrak{A}-\mathfrak{B}|}{1-|\mathfrak{A}|}$. We obtain

$$\| T_f \| \leq \min\left(\frac{2(\mathfrak{A}-\mathfrak{B})(\mathfrak{A}+2)}{\mathfrak{A}+1}, \frac{2|\mathfrak{A}-\mathfrak{B}|}{1-|\mathfrak{A}|}\right) = \frac{2(\mathfrak{A}-\mathfrak{B})}{1+\mathfrak{A}}.$$

□

Remark 3.1. *If we choose $\mathfrak{A} = 1-2\alpha$ and $\mathfrak{B}=-1$, then our result reduces to that of Mahzoon et. al [8].*

3.2. **Norm estimation for the subclass $\mathfrak{V}^*{}_\sigma[\mathfrak{A},\mathfrak{B}]$.** In this subsection we give some remarks on Theorem 1.2, [10]. Let $f \in \mathfrak{V}^*{}_\sigma[\mathfrak{A},\mathfrak{B}]$. Then the conditions (2.2) and (2.3) hold. From the geometric meaning of the function $\frac{1+\mathfrak{A}z}{1+\mathfrak{B}z}$, subordination priniciple and (2.2) we have

$$\frac{z^2 f'(z)}{[f(z)]^2} \prec \frac{1+\mathfrak{A}z}{1+\mathfrak{B}z}.$$

Due to subordination, a Schwarz function φ can be found so that

$$\frac{z^2 f'(z)}{[f(z)]^2} = \frac{1+\mathfrak{A}\varphi(z)}{1+\mathfrak{B}\varphi(z)}. \tag{3.5}$$

The logarithmic differentiation of (3.5), gives

$$\frac{f''(z)}{f'(z)} = 2\left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right) + \left(\frac{\mathfrak{A}}{1+\mathfrak{A}\varphi(z)} + \frac{\mathfrak{B}}{1+\mathfrak{B}\varphi(z)}\right)\varphi'(z). \tag{3.6}$$

Again, from (3.5) we get

$$\frac{f'(z)}{f(z)} = \frac{f(z)}{z}\left(\frac{1+\mathfrak{A}\varphi(z)}{z(1+\mathfrak{B}\varphi(z))}\right). \tag{3.7}$$

Using (3.7) in (3.6), we have

$$\frac{f''(z)}{f'(z)} = 2\left(\frac{f(z)}{z}\left(\frac{1+\mathfrak{A}\varphi(z)}{z(1+\mathfrak{B}\varphi(z))}\right) - \frac{1}{z}\right) + \left(\frac{\mathfrak{A}}{1+\mathfrak{A}\varphi(z)} + \frac{\mathfrak{B}}{1+\mathfrak{B}\varphi(z)}\right)\varphi'(z). \tag{3.8}$$

To simplify the above relation (3.8), we need to estimate $\left|\frac{f(z)}{z}\right|$, which remains as an open question so far. Also, we remark that Shalini et. al [10], have used the relation

$$\frac{f'(z)}{f(z)} = \frac{1+\mathfrak{A}\varphi(z)}{z(1+\mathfrak{B}\varphi(z))},$$

in the proof of Theorem 1.2 by mistake and this means that f belongs to both $\mathcal{S}^*[\mathfrak{A},\mathfrak{B}]$ and $\mathfrak{V}^*[\mathfrak{A},\mathfrak{B}]$.

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