SUM OF SOFT TOPOLOGICAL ORDERED SPACES

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ABSTRACT. This study aims to introduce the concept of sum of soft topological ordered spaces using pairwise disjoint soft topological ordered spaces and investigate its main properties. To link between soft topological spaces and their sum, we define ordered additive, finitely ordered additive and countably ordered additive properties. Then, we demonstrate that the properties of being p-soft T_i-ordered, soft T_i-ordered and strong soft T_i-ordered spaces are ordered additive and prove that the properties of monotonically soft compact and ordered soft compact spaces are finitely ordered additive. In this content, we study some features of soft λ-continuous, soft λ-open, soft λ-closed and soft λ-homeomorphism, where $\lambda \in \{I, D, B\}$. Finally, we give some examples and discuss under which conditions a soft topological ordered space represents the sum of some soft ordered topological spaces.

1. INTRODUCTION

Soft sets have became a very popular tool in dealing with problems contain uncertainty and vague since it was introduced by Molodtsov [31] in 1999. Soft set is an emerging field of research which has attracted many researchers. Its merits compared with probability theory and fuzzy set theory to overcome uncertainty and its applications in different fields were demonstrated in [32]. In 2003, Maji et al. [30] defined some operations between two soft sets such as

soft union, intersection and equality. Ali et al. [1] did some amendments for some results obtained by [30]. Also, they established new types of soft union and intersection between two soft sets. El-Shafei et al. [23] introduced the relations of partial belong and total non-belong and studied their main properties. Some applications of these relations in terms of soft axioms and decision making problems were given in [10, 24]. Recently, Al-shami [2] has revised some results of gf-soft equality relations and then Al-shami and El-Shafei [12] have defined new type of soft equality relation and initiated new types of soft linear equations with respect to some soft equality relations.

Shabir and Naz [34] employed soft sets to define soft topological spaces over an initial universal set with a fixed set of parameters. A lot of researches have been done to find out the different structures and concepts of soft topology similarly to their counterparts on classical topology. Aygünoğlu and Aygün [21] introduced the concept of soft compactness and showed its main features with the help of examples. Then, Hida [27] explored another type of soft compactness which represents one of divergences between soft topology and classical topology. Al-shami et al. [13] defined weak types of soft compact spaces, namely almost soft compact and mildly soft compact spaces. Soft compactness was generalized by using soft α-open, soft pre-open and soft semi-open sets in [6, 7, 15]. Another celebrated concept is soft connectedness and soft paracompactness which were probed in [29]. The authors of [28] defined mappings in soft setting and established main properties. Then the authors of [35] introduced and discussed the concepts of soft continuous and homeomorphism mappings. To preserve some properties between soft topology and classical topology, Al-shami and Kočinac [19] investigated the sufficient conditions of the interchangeability of soft interior and soft closure operators between soft topological space and their parametric topological spaces.

Al-shami et al. [16] explored the concepts of monotonically soft sets and soft topological ordered spaces. Then, Al-shami and El-Safei [8] introduced supra soft topological ordered spaces and shed light on supra ordered soft separation axioms. Also, they [11] studied two types of ordered soft separation axioms with various illustrative examples. Al-shami et al. [14] defined some types of soft ordered mappings and then these types were generalized by using some famous generalizations of soft open sets in [9, 17, 18, 22]. Recently, I have introduced

and studied soft compact and soft connected spaces on ordered setting in [4] and [5], respectively.

This paper is one of the outcomes of future works mentioned in [20]. The main object of this study is to introduce the concept of sum of soft topological ordered spaces using pairwise disjoint soft topological ordered spaces. Our results mainly investigate invariant properties between soft topological ordered spaces and their sum. We classify these properties to ordered additive, finitely ordered additive and countably ordered additive properties. In the end, we prove some results related to generate sum of soft topological ordered spaces.

2. PRELIMINARIES

To make our paper self contained, we recall some definitions and results that given in some previous studies.

2.1. **Soft sets.** Throughout the paper Y will be a nonempty set, called an *initial universal set*, 2^Y its power set, A a nonempty set, called the *set of parameters*.

Definition 2.1. *[31] A* soft set *over Y is an ordered pair (ξ, A) such that A is a set of parameters and ξ is a mapping of A into 2^Y.*

We often write a soft set (ξ, A) as

$$(\xi, A) = \{(a, \xi(a)) : a \in A \text{ and } \xi(a) \in 2^Y\}.$$

Through this work, the collection of all soft sets over Y under a set of parameters A is denoted by $\mathsf{SS}(Y_A)$. Also, we use the different notations (ω, B), (δ, C), (η, D) for soft sets.

Definition 2.2. *[23, 34] Let (ξ, A) be a soft set over Y and $y \in Y$. We write:*
  (1) *$y \in (\xi, A)$ if $y \in \xi(a)$ for some $a \in A$; and $y \notin (\xi, A)$ if $y \notin \xi(a)$ for every $a \in A$.*
  (2) *$y \in (\xi, A)$ if $y \in \xi(a)$ for every $a \in A$; and $y \notin (\xi, A)$ if $y \notin \xi(a)$ for some $a \in A$. In particular, $y \in \widetilde{Y}$ means $y \in Y$.*

Definition 2.3. *[30] A soft set (ξ, A) over Y is said to be the* null soft set *if $\xi(a) = \emptyset$ for all $a \in A$; and it is said to be the* absolute soft set *if $\xi(a) = Y$ for all $a \in A$.*
  The null and absolute soft sets are denoted by $\widetilde{\emptyset}_Y$ and $\widetilde{Y}$, respectively.

Definition 2.4. *[26] (ξ, A) is called a* soft subset *of (ω, B), denoted by $(\xi, A)\widetilde{\subseteq}(\omega, B)$, if A is a subset of B and $\xi(a)$ is a subset of $\omega(a)$ for all $a \in A$. The two soft sets are called* soft equal *if each of them is a soft subset of the other.*

Definition 2.5. *[1] The* relative complement *of (ξ, A) is a soft set $(\xi, A)^c = (\xi^c, A)$ such that the map $\xi^c : A \to 2^Y$ is defined by $\xi^c(a) = Y \setminus \xi(a)$ for each $a \in A$.*

Definition 2.6. *[30] The* union *of soft sets (ξ, A) and (ω, B) over Y, denoted by $(\xi, A)\widetilde{\bigcup}(\omega, B)$, is the soft set (δ, C), where $C = A \bigcup B$ and $\delta : C \to 2^Y$ is a mapping defined by*

$$
\delta(c) = \begin{cases}
\xi(c) & : & c \in A \setminus B \\
\omega(c) & : & c \in B \setminus A \\
\xi(c) \bigcup \omega(c) & : & c \in A \bigcap B.
\end{cases}
$$

Definition 2.7. *[1] The* intersection *of soft sets (ξ, A) and (ω, B) over Y, denoted by $(\xi, A)\widetilde{\bigcap}(\omega, B)$, is a soft set (δ, C), where $C = A \bigcap B \neq \emptyset$ and $\delta : C \to 2^Y$ is a mapping defined by $\delta(c) = \xi(c) \bigcap \omega(c)$.*

The operators of union and intersection were generalized for an arbitrary family of soft sets over a common universe Y and with a common set of parameters.

Definition 2.8. *[25, 33] The* union *of a family $\{(\xi_i, A) : i \in I\}$ of soft sets over the common universe Y, denoted by $\widetilde{\bigcup}_{i \in I}(\xi_i, A)$, is the soft set (η, A), where for each $a \in A$, $\eta(a) = \bigcup_{i \in I} \xi_i(a)$.*
The intersection *of a family $\{(\xi_i, A); i \in I\}$ over the common universe Y, denoted by $\widetilde{\bigcap}_{i \in I}(\xi_i, A)$, is the soft set (η, A), where for each $a \in A$, $\eta(a) = \bigcap_{i \in I} \xi_i(a)$.*

Definition 2.9. *A collection of two or more soft sets is said to be* pairwise disjoint *if the intersection of any two distinct soft sets is the null soft set.*

Definition 2.10. *[28] A* soft mapping *between $\mathsf{SS}(Y_A)$ and $\mathsf{SS}(Z_A)$ is a mapping $f : Y \to Z$ such that the image of $(\xi, A) \in \mathsf{SS}(Y_A)$ and preimage of $(\theta, A) \in \mathsf{SS}(Z_A)$ are defined by:*

  (i) $f(\xi, A) = (f(\xi), A)$, *where* $[f(\xi)](a) = f(\xi(a))$, $\quad a \in A$;
  (ii) $f^{\leftarrow}(\theta, A) = (f^{\leftarrow}(\theta), A)$, *where* $[f^{\leftarrow}(\theta)](a) = f^{\leftarrow}(\theta(a))$, $\quad a \in A$.

Definition 2.11. *[14] $(Y, A, \preceq)$ is said to be a* partially ordered soft set *on $Y \neq \emptyset$ if $(Y, \preceq)$ is a partially ordered set.*

Definition 2.12. *[14] An* increasing *operator i and a* decreasing *operator d are two soft maps of $(\mathsf{SS}(Y_A), \preceq)$ into $(\mathsf{SS}(Y_A), \preceq)$ defined as follows: for each soft subset (ξ, A) of $\mathsf{SS}(Y_A)$*

    (1) *$i(\xi, A) = (i\xi, A)$, where $i\xi$ is a mapping of A into 2^Y given by $i\xi(a) = i(\xi(a)) = \{y \in Y : x \preceq y \text{ for some } x \in \xi(a)\}$;*

    (2) *$d(\xi, A) = (d\xi, A)$, where $d\xi$ is a mapping of A into 2^Y given by $d\xi(a) = d(\xi(a)) = \{x \in Y : y \preceq x \text{ for some } x \in \xi(a)\}$.*

Definition 2.13. *[14] A soft subset (ξ, A) of $(Y, A, \preceq)$ is said to be increasing (resp. decreasing) provided that $(\xi, A) = i(\xi, A)$ (resp. $(\xi, A) = d(\xi, A)$).*

2.2. **Soft topology.**

Definition 2.14. *[34] The family τ of soft sets over Y under a set of parameters A is called a* soft topology *on Y provided that it is closed under arbitrary union and finite intersection and contains $\widetilde{Y}$ and $\widetilde{\emptyset}_Y$.*

    The triple (Y, τ, A) is called a soft topological space*. An element (ξ, A) is called a* soft open set *(resp.* soft closed set*) if (ξ, A) (resp. $(\xi, A)^c$) belongs to τ.*

Definition 2.15. *[14] For a subset $Z \neq \emptyset$ of (Y, τ, A), the family $\tau_Z = \{\widetilde{Z} \widetilde{\bigcap} (\xi, A) : (\xi, A) \in \tau\}$ is called a* soft relative topology *on Z and the triple (Z, τ_Z, A) is called a* soft subspace *of (Y, τ, A).*

Definition 2.16. *[23] A soft set (ξ, A) over Y is called* stable *provided that there is $S \subseteq Y$ such that $\xi(a) = S$ for each $a \in A$; a soft topological space (Y, τ, A) is called* stable *provided that all proper non null soft open sets are stable.*

Definition 2.17. *[14] A quadrable system $(Y, \tau, A, \preceq)$ is said to be a* soft topological ordered space*, where (Y, τ, A) is a* soft topological space *and $(Y, \preceq)$ is a partially ordered set.*

Definition 2.18. *[14] $(Y, \tau, A, \preceq)$ is said to be:*

    (1) Lower *p-soft T_0-ordered if for every $y \npreceq z \in Y$, there is an increasing soft open set (ξ, A) such that $y \in (\xi, A)$ and $z \notin (\xi, A)$.*

    (2) Upper *p-soft T_0-ordered if for every $y \npreceq z \in Y$, there is a decreasing soft open set (ξ, A) such that $z \in (\xi, A)$ and $y \notin (\xi, A)$.*

    (3) *p-soft T_0-ordered if it is lower p-soft T_0-ordered or upper p-soft T_0-ordered.*

    (4) *p-soft T_1-ordered if it is lower p-soft T_0-ordered and upper p-soft T_0-ordered.*

(5) p-soft T_2-ordered *if for every $y \npreceq z \in Y$, there are disjoint increasing soft open set (ξ, A) and decreasing soft open set (ω, A) such that $y \in (\xi, A)$ and $z \in (\omega, A)$.*

(6) p-soft regular ordered *if for every increasing (resp. decreasing) soft closed set (η, A) and $y \in Y$ such that $y \notin (\eta, A)$, there exist disjoint increasing (resp. decreasing) soft open set (ξ, A) and decreasing (resp. increasing) soft open set (ω, A) such that $(\eta, A)\widetilde{\subseteq}(\xi, A)$ and $y \in (\omega, A)$.*

(7) soft normal ordered *if for every two disjoint soft closed sets (η_1, A) and (η_2, A) such that (η_1, A) is increasing and (η_2, A) is decreasing, there exist two disjoint increasing soft open set (ξ, A) and decreasing soft open set (ω, A) such that $(\eta_1, A)\widetilde{\subseteq}(\xi, A)$ and $(\eta_2, A)\widetilde{\subseteq}(\omega, A)$.*

(8) p-soft T_3-ordered *(resp.* p-soft T_4-ordered*) if it is both* p-soft T_1-ordered *and* p-soft regular ordered *(resp. soft normal ordered).*

If we replace $\nsubseteq$ by $\notin$ in the above definition, we obtain *strong soft T_i-ordered spaces* given in [11]; and if we replace $\nsubseteq$ by $\notin$ and soft open by soft neighborhood we obtain *soft T_i-ordered spaces* given in [11] as well.

Proposition 2.1. *[20] Let $\{(Y_i, \tau_i, A) : i \in I\}$ be a family of pairwise disjoint soft topological spaces and $Y = \bigcup_{i \in I} Y_i$. Then the collection*

$$\tau = \{(\xi, A) \text{ over } \bigcup_{i \in I} \widetilde{Y}_i : (\xi, A)\widetilde{\bigcap}\widetilde{Y}_i = \{(a, \xi(a) \bigcap Y_i) : a \in A\} \text{ is a soft open set in} \\ (Y_i, \tau_i, A) \text{ for every } i \in I\}$$

defines a soft topology on Y with a fixed set of parameters A.

This soft topological space is called sum of soft topological spaces *and denoted by $(\oplus_{i \in I} Y_i, \tau, A)$.*

Proposition 2.2. *[20] All soft sets $\widetilde{Y}_i$ are soft clopen in $(\oplus_{i \in I} Y_i, \tau, A)$.*

Definition 2.19. *[20] A subset (ξ, A) of $(Y, A, \preceq)$ is called a* monotonically soft set *if (ξ, A) is increasing or decreasing. In other words, $(\xi, A) = i(\xi, A)$ or $(\xi, A) = d(\xi, A)$.*

Definition 2.20. *[20] The collection $\{(\xi_j, A) : j \in J\}$ of soft open subsets of $(Y, \tau, A, \preceq)$ is called a* monotonically soft open cover *of $\widetilde{Y}$ provided that $\widetilde{Y} = \bigcup_{j \in J}(\xi_j, A)$ and all (ξ_j, A) are monotonic.*

Definition 2.21. *[20] $(Y, \tau, A, \preceq)$ is said to be:*

(1) *monotonically soft compact provided that every monotonically soft open cover of $\widetilde{Y}$ has a finite subcover.*

(2) *ordered soft compact if every soft open cover of $\widetilde{Y}$ has a finitely monotonic subcover.*

Proposition 2.3. *[20] Every monotonically closed (resp. closed) subset F of a monotonically compact (resp. an ordered compact) space $(Y, \tau, A, \preceq)$ is monotonically compact (resp. ordered compact).*

Definition 2.22. *[20] $(Y, \tau, A, \preceq)$ is said to be:*

(1) *monotonically soft connected if $\widetilde{\emptyset}$ and $\widetilde{Y}$ are the only monotonically soft clopen.*

(2) *monotonically soft hyperconnected if every monotonically soft open set is monotonically soft dense.*

Definition 2.23. *[14] $f_\phi : (Y, \tau, A, \preceq_1) \to (Z, \theta, A, \leq)$ is called:*

(i) *soft I (resp. soft D, soft B) -continuous if and only if the inverse image of each soft open set is soft I (resp. soft D, soft B) -open.*

(ii) *soft I (resp. soft D, Soft B) -open if the image of every soft open is soft I (resp. soft D, soft B) -open.*

(iii) *soft I (resp. soft D, Soft B) -closed if the image of every soft closed is soft I (resp. soft D, soft B) -closed.*

Definition 2.24. *[14] A bijective soft map $g_\phi : (Y, \tau, A, \preceq_1) \to (Z, \theta, A, \leq)$ is called soft I (resp. soft D, soft B) -homeomorphism if it is soft I-continuous and soft I-open (resp. soft D-continuous and soft D-open, soft B-continuous and soft B-open).*

3. SUM OF SOFT TOPOLOGICAL ORDERED SPACES

In this section, we present the concept of sum of soft topological spaces on ordered setting and then we define ordered additive, finitely ordered additive and countably ordered additive properties. We discuss some concepts in terms of these properties with the help of illustrative examples.

Proposition 3.1. *Let $\{(Y_i, \tau_i, A, \preceq_i) : i \in I\}$ be a family of pairwise disjoint soft topological ordered spaces. Then $(Y, \tau, A, \preceq)$ is a soft topological ordered space, where*

(1) $Y = \bigcup_{i \in I} Y_i$;
(2) $\tau = \{(\xi, A) \text{ over } \bigcup_{i \in I} \widetilde{Y}_i : (\xi, A) \widetilde{\bigcap} \widetilde{Y}_i = \{(a, \xi(a) \bigcap Y_i) : a \in A\} \in \tau_i \text{ for every } i \in I\}$ and
(3) $\preceq = \bigcup_{i \in I} \preceq_i$.

Proof. It follows from Proposition 2.1 that (Y_i, τ_i, A) is a soft topological space. It remains to prove that $\preceq$ is a partial order relation on Y. It is clear that $\preceq$ is reflexive on Y. Since $\preceq_i$ is anti-symmetric and transitive for each i and $\preceq_i \bigcap \preceq_j = \emptyset$ for each $i \neq j$, then $\preceq$ is anti-symmetric and transitive on Y. Thus, $\preceq$ is a partial order relation on Y. Hence, $(Y, \tau, A, \preceq)$ is a soft topological ordered space. $\square$

Definition 3.1. The soft topological ordered space $(Y, \tau, A, \preceq)$ given in the above proposition is said to be the *sum of soft topological ordered spaces* and is denoted by $(\oplus_{i \in I} Y_i, \tau, A, \preceq)$.

Proposition 3.2. *If (η, A) is a monotonic soft subset of $(\oplus_{i \in I} Y_i, A, \preceq)$, then $(\eta, A) \widetilde{\bigcap} \widetilde{Y}_i$ is a monotonic subset of $(Y_i, A, \preceq_i)$ for every $i \in I$.*

Proof. Necessity: Let (η, A) be a monotonically soft subset of $(\oplus_{i \in I} Y_i, A, \preceq)$. Say, it is increasing. Suppose that there exists $i \in I$ such that $(\eta, A) \widetilde{\bigcap} \widetilde{Y}_i$ is not an increasing soft subset of $(Y_i, A, \preceq_i)$. Then there exist $a \in A$ and $y \in Y_i$ such that $y \notin \eta(a) \bigcap Y_i$ and $x \preceq_i y$ for some $x \in \eta(a) \bigcap Y_i$. Since $Y_i \bigcap Y_j = \emptyset$ for each $i \neq j$, then (η, A) is not increasing. This is a contradiction. $\square$

The converse of the above proposition is not always true as it is illustrated in the following example.

Example 1. *Let $A = \{a_1, a_2\}$, $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$. Consider $\preceq_1 = \triangle \bigcup \{(x_1, x_2)\}$ and $\preceq_2 = \triangle \bigcup \{(y_2, y_1)\}$ are partial order relations on X and Y, respectively. Now, $(\eta, A) = \{(a_1, \{x_1, y_1\}), (a_2, \{x_1, y_1\})\}$ is neither decreasing nor increasing soft subset of $(X \bigcup Y, A, \preceq)$. Therefore it is not monotonic. However, $(\eta, A) \widetilde{\bigcap} \widetilde{X} = \{(a_1, \{x_1\}), (a_2, \{x_1\})\}$ is a decreasing soft subset of $(X, A, \preceq_1)$ and $(\eta, A) \widetilde{\bigcap} \widetilde{Y} = \{(a_1, \{y_1\}), (a_2, \{y_1\})\}$ is an increasing soft subset of $(Y, A, \preceq_2)$.*

Proposition 3.3. *A soft subset (η, A) of $(\oplus_{i \in I} Y_i, A, \preceq)$ is increasing (resp. deceasing) if and only if all soft sets $(\eta, A) \widetilde{\bigcap} \widetilde{Y}_i$ are increasing (resp. deceasing) in $(Y_i, A, \preceq_i)$.*

Proof. The necessary condition follows from Proposition 3.2.

To prove the sufficient condition, let $(\eta, A)\widetilde{\bigcap}\widetilde{Y_i}$ be an increasing (resp. a deceasing) soft subset of $(Y_i, A, \preceq_i)$ for every $i \in I$. Since $Y_i \bigcap Y_j = \emptyset$ for each $i \neq j$ and $\preceq = \bigcup_{i \in I} \preceq_i$, then (η, A) is an increasing (resp. a deceasing) soft subset of $(\oplus_{i \in I} Y_i, A, \preceq)$. □

Proposition 3.4. *All soft sets $\widetilde{Y_i}$ are monotonically soft clopen in $(\oplus_{i \in I} Y_i, \tau, A, \preceq)$.*

Proof. It follows from Proposition 2.2 and Proposition 3.3. □

Corollary 3.1. *Every sum of soft topological spaces is monotonically soft disconnected.*

Proposition 3.5. *If $\{(Y_i, \tau_i, A, \preceq_i) : i \in I\}$ is a class of pairwise disjoint soft topological ordered spaces and X_i is a subspace of Y_i for every $i \in I$, then the soft ordered topology of the sum of subspaces $\{(X_i, \tau_{X_i}, A, \preceq_{X_i}) : i \in I\}$ and the soft topological ordered subspace on $\bigcup_{i \in I} X_i$ of the sum soft topology $(\oplus_{i \in I} Y_i, \tau, A, \preceq)$ coincide.*

Proof. Straightforward. □

Definition 3.2. A property $\mathcal{P}$ is said to be:

(1) *ordered additive* if for any family of soft topological ordered spaces $\{(Y_i, \tau_i, A, \preceq_i) : i \in I\}$ with the property $\mathcal{P}$, the sum of this family also has property $\mathcal{P}$.

(2) *finitely ordered additive* (resp., *countably ordered additive*) if for any finite (resp., countable) family soft topological ordered spaces with the property $\mathcal{P}$, the sum of this family also has property $\mathcal{P}$.

Theorem 3.1. *The property of being a p-soft T_i-ordered space is an ordered additive property for $i = 0, 1, 2, 3, 4$.*

Proof. We prove the theorem in the case of $i = 2$. Let $y \npreceq z \in \oplus_{i \in I} Y_i$. Then we have the following two cases:

1. There exists $i_0 \in I$ such that $y, z \in Y_{i_0}$.

Since $(Y_{i_0}, \tau_{i_0}, A, \preceq_{i_0})$ is p-soft T_2-ordered, then there exist disjoint increasing soft open set (ξ, A) and decreasing soft open set (ω, A) in $(Y_{i_0}, \tau_{i_0}, A, \preceq_{i_0})$ such that $y \in (\xi, A)$ and $z \in (\omega, A)$. It follows from Proposition 3.3 that (ξ, A) is

increasing soft open set and (ω, A) is decreasing soft open set in $(\oplus_{i \in I} Y_i, \tau, A, \preceq)$ such that $(\xi, A) \widetilde{\bigcap}(\omega, A) = \widetilde{\emptyset}$.

2. There exist $i_0 \neq j_0 \in I$ such that $y \in Y_{i_0}$ and $z \in Y_{j_0}$.

Now, $\widetilde{Y_{i_0}}$ is an increasing soft open subset of $(Y_{i_0}, \tau_{i_0}, A, \preceq_{i_0})$ and $\widetilde{Y_{j_0}}$ is a decreasing soft open subset of $(Y_{j_0}, \tau_{j_0}, A, \preceq_{i_0})$. Obviously, $\widetilde{Y_{i_0}}$ and $\widetilde{Y_{j_0}}$ are disjoint soft open subsets of $(\oplus_{i \in I} Y_i, \tau, A, \preceq)$ such that $\widetilde{Y_{i_0}}$ is increasing and $\widetilde{Y_{j_0}}$ is decreasing.

It follows from the two cases above that $(\oplus_{i \in I} Y_i, \tau, A, \preceq)$ is a p-soft T_2-ordered space.

The theorem can be proved similarly in the cases of $i = 0, 1$.

To prove the theorem in the cases of $i = 3$ and $i = 4$, it suffices to prove the p-soft regularity ordered and soft normality ordered, respectively.

First, we prove the p-soft regularity ordered property. Let (η, A) be an increasing soft closed subset of $(\oplus_{i \in I} Y_i, \tau, A, \preceq)$ such that $y \notin (\eta, A)$. It follows from Proposition 3.2, that $(\eta, A) \widetilde{\bigcap} \widetilde{Y_i}$ is increasing soft closed in $(Y_i, \tau_i, A, \preceq_i)$ for each $i \in I$. $y \in \oplus_{i \in I} Y_i$ implies that there is only $i_0 \in I$ such that $y \in Y_{i_0}$. Therefore there are disjoint soft open subsets (ξ, A) and (ω, A) of $(Y_{i_0}, \tau_{i_0}, A, \preceq_{i_0})$ containing $(\eta, A) \widetilde{\bigcap} \widetilde{Y_{i_0}}$ and y such that (ξ, A) is increasing and (ω, A) is decreasing. Now, $(\xi, A) \underset{i \neq i_0}{\widetilde{\bigcup}} \widetilde{Y_i}$ is an increasing soft open subset of $(\oplus_{i \in I} Y_i, \tau, A, \preceq)$ containing (η, A). The disjointness between $(\xi, A) \bigcup_{i \neq i_0} Y_i$ and (ω, A) ends the proof that $(\oplus_{i \in I} Y_i, \tau, A, \preceq)$ is a p-soft regular ordered space.

Second, we prove the soft normality ordered property. Let (η, A) and (δ, A) be two disjoint soft closed subsets of $(\oplus_{i \in I} Y_i, \tau, A, \preceq)$ such that (η, A) is increasing and (δ, A) is decreasing. It follows from Proposition 3.3, that $(\eta, A) \widetilde{\bigcap} \widetilde{Y_i}$ and $(\delta, A) \widetilde{\bigcap} \widetilde{Y_i}$ are soft closed in $(Y_i, \tau_i, A, \preceq_i)$ for each $i \in I$ such that $(\eta, A) \widetilde{\bigcap} \widetilde{Y_i}$ is increasing and $(\delta, A) \widetilde{\bigcap} \widetilde{Y_i}$ is decreasing. Since $(Y_i, \tau_i, A, \preceq_i)$ is soft normal ordered for each $i \in I$, then there exist disjoint increasing soft open set (ξ_i, A) and decreasing soft open set (ω_i, A) in $(Y_i, \tau_i, A, \preceq_i)$ such that $(\eta, A) \widetilde{\bigcap} \widetilde{Y_i} \widetilde{\subseteq} (\xi_i, A)$ and $(\delta, A) \widetilde{\bigcap} \widetilde{Y_i} \widetilde{\subseteq} (\omega_i, A)$. Therefore $(\eta, A) \widetilde{\subseteq} \underset{i \in I}{\widetilde{\bigcup}} (\xi_i, A)$, $(\delta, A) \widetilde{\subseteq} \underset{i \in I}{\widetilde{\bigcup}} (\omega_i, A)$ and $[\underset{i \in I}{\widetilde{\bigcup}} (\xi_i, A)] \widetilde{\bigcap} [\underset{i \in I}{\widetilde{\bigcup}} (\omega_i, A)] = \widetilde{\emptyset}$. Hence, $(\oplus_{i \in I} Y_i, \tau, A, \preceq)$ is a soft normal ordered space. $\qquad \square$

Theorem 3.2. *The property of being a soft T_i-ordered (strong soft T_i-ordered) space is an ordered additive property for $i = 0, 1, 2, 3, 4$.*

Proof. The proof is similar to that of Theorem 3.1.   □

Proposition 3.6. *The property of being a monotonically soft compact space is a finitely ordered additive property.*

Proof. Let $\{(Y_k, \tau_k, A, \preceq_k) : k \in \{1, 2, \ldots, n\}\}$ be a finite family of pairwise disjoint monotonically soft compact spaces and let $(\oplus_{k=1}^n Y_k, \tau, A, \preceq)$ be the sum of this family. Suppose that $\{(\xi_i, A) : i \in I\}$ is a soft open cover of $\widetilde{Y} = \widetilde{\bigcup}_{k=1}^n \widetilde{Y}_k$. Then $\widetilde{Y}_k = \widetilde{\bigcup}_{i \in I}[(\xi_i, A) \widetilde{\bigcap} \widetilde{Y}_k]$ for every $k \leq n$. Since $(Y_k, \tau_k, A, \preceq_k)$ is monotonically soft compact for every $k \leq n$, then there exist finite subsets $M_1, M_2, \ldots, M_n$ of I such that $\widetilde{Y}_1 = \widetilde{\bigcup}_{i \in M_1}[(\xi_i, A) \widetilde{\bigcap} \widetilde{Y}_1]$ and all $(\xi_i, A) \widetilde{\bigcap} \widetilde{Y}_1$ are monotonic, $\widetilde{Y}_2 = \widetilde{\bigcup}_{i \in M_2}[(\xi_i, A) \widetilde{\bigcap} \widetilde{Y}_2]$ and all $(\xi_i, A) \widetilde{\bigcap} \widetilde{Y}_2$ are monotonic, $\ldots$, $\widetilde{Y}_n = \widetilde{\bigcup}_{i \in M_n}[(\xi_i, A) \widetilde{\bigcap} \widetilde{Y}_n]$ and all $(\xi_i, A) \widetilde{\bigcap} \widetilde{Y}_n$ are monotonic. Letting $M = \bigcup_{k=1}^n M_k$. Now, $\widetilde{Y} = \widetilde{\bigcup}_{i \in M}[(\xi_i, A) \widetilde{\bigcap} \widetilde{Y}_k]$ for every $k \leq n$. Since M is finite and all $(\xi_i, A) \widetilde{\bigcap} \widetilde{Y}_k$ are monotonic, then $(\oplus_{k=1}^n Y_s, \tau, A, \preceq)$ is monotonically soft compact.   □

Proposition 3.7. *The property of being an ordered soft compact space is a finitely ordered additive property.*

Proof. The proof is similar to that of Proposition 3.6.   □

The following example shows that the properties of monotonically soft compactness and ordered soft compactness are not ordered additive.

Example 2. *Let $A = \{a_1, a_2\}$ and let $Y_n = \{2n-1, 2n\}$, where n belongs to the set of natural numbers $\mathbb{N}$. Consider τ_n is the discrete soft topology and $\preceq_n$ is the equality relation on Y_n for each n. Now, $\{(Y_n, \tau_n, A, \preceq_n) : n \in \mathcal{N}\}$ is a family of pairwise disjoint monotonically soft compact and ordered soft compact spaces. Obviously, $\preceq$ is the equality relation on Y and the sum of these soft spaces $(\oplus_{n \in \mathcal{N}} Y_n, \tau, A, \preceq)$ is soft discrete. Then $(\oplus_{n \in \mathcal{N}} Y_n, \tau, A, \preceq)$ is neither monotonically soft compactness neither ordered soft compactness. Hence, the properties of monotonically soft compactness and ordered soft compactness are not ordered additive.*

Proposition 3.8. *If the sum of soft topological ordered spaces $(\oplus_{i \in I} Y_i, \tau, A, \preceq)$ is monotonically soft compact (resp. ordered soft compact), then the following two assertions are true:*

(1) *all $(Y_i, \tau_i, A, \preceq_i)$ are monotonically soft compact (resp. ordered soft compact).*

(2) *the index set I is finite.*

Proof. 1. It follows from Corollary 3.4, $(Y_i, \tau_i, A, \preceq_i)$ is a monotonically soft closed subspace of $(\oplus_{i \in I} Y_i, \tau, A, \preceq)$ for each $i \in I$. It follows from Proposition 2.3 that $(Y_i, \tau_i, A, \preceq_i)$ is monotonically soft compact (ordered soft compact) for each $i \in I$.

2. Let $(\oplus_{i \in I} Y_i, \tau, A, \preceq)$ be the sum of soft topological ordered spaces. Then $\Lambda = \{Y_i : i \in I\}$ is a (monotonically) soft open cover of $\widetilde{Y} = \widetilde{\bigcup_{i \in I}} \widetilde{Y_i}$. It is clear that Λ does not have a finite subcover if I is infinite. This contradicts the fact that $(\oplus_{i \in I} Y_i, \tau, A, \preceq)$ is ordered (monotonically) soft compact. Hence, it must be that I is finite. $\square$

Similarly to the proof of Proposition 3.6, one can prove the following result.

Proposition 3.9. *The property of being a monotonically soft Lindelöf (ordered soft Lindelöf) space is a countably additive property.*

Definition 3.3. Let $\{f_i : (Y_i, \tau_i, A, \preceq_i) \to (Z_i, \theta_i, B, \leq_i) : i \in I\}$ be a family of soft mappings. Then we define a soft mapping $f : (\oplus_{i \in I} Y_i, \tau, A, \preceq) \to (\oplus_{i \in I} Z_i, \theta, B, \leq)$ as follows: For each soft subsets (ξ, A) and (ω, B) of $(\oplus_{i \in I} Y_i, \tau, A, \preceq)$ and $(\oplus_{i \in I} Z_i, \theta, B, \leq)$, respectively, we have:

(1) $f(\xi, A) = \widetilde{\bigcup_{i \in I}} f_i([(\xi, A) \widetilde{\bigcap} \widetilde{Y_i}])$; and

(2) $f^{-1}(\omega, B) = \widetilde{\bigcup_{i \in I}} f_i^{-1}([(\omega, B) \widetilde{\bigcap} \widetilde{Z_i}])$

Theorem 3.3. *A soft mapping $f : (\oplus_{i \in I} Y_i, \tau, A, \preceq) \to (\oplus_{i \in I} Z_i, \theta, B, \leq)$ is soft I (resp. soft D, soft B) -continuous if and only if every soft mappings $f_i : (Y_i, \tau_i, A, \preceq_i) \to (Z_i, \theta_i, B, \leq_i)$ is soft I (resp. soft D, soft B) -continuous.*

Proof. We prove the theorem in the case of soft I-continuity and one can prove the cases between parentheses similarly.

Necessity: Suppose that a soft mapping $f : (\oplus_{i \in I} Y_i, \tau, A, \preceq) \to (\oplus_{i \in I} Z_i, \theta, B, \leq)$ is soft I-continuous. Taking an arbitrary soft map $f_j : (Y_j, \tau_j, A, \preceq_j) \to (Z_j, \theta_j, B, \leq_j)$, where $j \in I$. Let (ξ, B) be a soft open subset of $(Z_j, \theta_j, B, \leq_i)$. Then (ξ, B) is a soft open subset of $(\oplus_{i \in I} Z_i, \theta, B, \leq)$. By assumption, $f^{-1}(\xi, B)$ is an increasing soft open subset of $(\oplus_{i \in I} Y_i, \tau, A, \preceq)$. Since $(\xi, B) \widetilde{\bigcap} \widetilde{Z_i} = \widetilde{\emptyset}_{Z_i}$ for each $i \neq j$, then

$f^{-1}(\xi, B) = f_j^{-1}(\xi, B)$. Therefore $f_j^{-1}(\xi, B)$ is an increasing soft open subset of $(Y_j, \tau_j, A, \preceq_j)$, as required.

Sufficiency: Suppose that $f_i : (Y_i, \tau_i, A, \preceq_i) \to (Z_i, \theta_i, B, \leq_i)$ is soft I-continuous for every $i \in I$ and let (ω, B) be a soft open subset of $(\oplus_{i \in I} Z_i, \theta, B, \leq)$. Now, $(\omega, B) \widetilde{\cap} \widetilde{Z}_i$ is a soft open subset of $(Z_i, \theta_i, B, \leq_i)$ for every $i \in I$. By assumption, $f_i^{-1}[(\omega, B) \widetilde{\cap} \widetilde{Z}_i]$ is an increasing soft open subset of $(Y_i, \theta_i, A, \preceq_i)$ for every $i \in I$. Therefore $\widetilde{\bigcup_{i \in I}} f_i^{-1}[(\omega, B) \widetilde{\cap} \widetilde{Z}_i]$ is an increasing soft open subset of $(\oplus_{i \in I} Y_i, \tau, A, \preceq)$. Since $f^{-1}(\omega, B) = \widetilde{\bigcup_{i \in I}} f_i^{-1}[(\omega, B) \widetilde{\cap} \widetilde{Z}_i]$, then $f^{-1}(\omega, B)$ is an increasing soft open subset of $(\oplus_{i \in I} Y_i, \tau, A, \preceq)$, as required. $\square$

In a similar way, one can prove the following three results.

Theorem 3.4. *A soft mapping* $f : (\oplus_{i \in I} Y_i, \tau, A, \preceq) \to (\oplus_{i \in I} Z_i, \theta, B, \leq)$ *is soft I (resp. soft D, soft B) -open if and only if every soft mappings* $f_i : (Y_i, \tau_i, A, \preceq_i) \to (Z_i, \theta_i, B, \leq_i)$ *is soft I (resp. soft D, soft B) -open.*

Theorem 3.5. *A soft mapping* $f : (\oplus_{i \in I} Y_i, \tau, A, \preceq) \to (\oplus_{i \in I} Z_i, \theta, B, \leq)$ *is soft I (resp. soft D, soft B) -closed if and only if every soft mappings* $f_i : (Y_i, \tau_i, A, \preceq_i) \to (Z_i, \theta_i, B, \leq_i)$ *is soft I (resp. soft D, soft B) -closed*

Corollary 3.2. *A soft mapping* $f : (\oplus_{i \in I} Y_i, \tau, A, \preceq) \to (\oplus_{i \in I} Z_i, \theta, B, \leq)$ *is soft I (resp. soft D, soft B) -homeomorphism if and only if every soft mappings* $f_i : (Y_i, \tau_i, A, \preceq_i) \to (Z_i, \theta_i, B, \leq_i)$ *is soft I (resp. soft D, soft B) -homeomorphism.*

The other path of this study is the answer of the following two questions:

(1) Under what conditions a soft topological ordered space represents the sum of soft topological ordered spaces?
(2) If a soft topological ordered space represents the sum of soft topological ordered spaces, what is the maximum number of these soft topological ordered spaces?

The following results answer these questions.

Theorem 3.6. *If* $(Y, \tau, A, \preceq)$ *is stable monotonically soft disconnected, then it represents the sum of two soft topological ordered spaces.*

Proof. Since $(Y, \tau, A, \preceq)$ is monotonically soft disconnected, then it contains at least a proper monotonically soft clopen set (ξ, A). Since $(Y, \tau, A, \preceq)$ is stable,

then $\xi(a) = X \subseteq Y$ for each $a \in A$. So that the two soft subspaces $(X, \tau_X, A, \preceq_X)$ and $(X^c, \tau_{X^c}, A, \preceq_{X^c})$ are soft topological ordered spaces such that $(Y, \tau, A, \preceq)$ is their sum. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\Box$

Corollary 3.3. *If* $(Y, \tau, A, \preceq)$ *contains* m *stable monotonically soft clopen sets, then* $(Y, \tau, A, \preceq)$ *represents the sum of* $2m$ *soft topological ordered spaces.*

4. Conclusion

Soft topological ordered space combines of two mathematical structures: soft topology and partial order relation on the universal set. Both of soft topology and partial order relation are defined as independent each of other. However, the interaction between them occurs in the case of defining some concepts using some characteristics of soft topology and partial order relation such as increasing (decreasing) soft open and soft closed sets. In this article, we have introduced and studied the concepts of soft topological ordered spaces. We have defined ordered additive, finitely ordered additive and countably ordered additive properties and then investigated some concepts with respect to these properties. It should be noted that the results obtained herein are genuine generalizations of results obtained in [3].

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