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FIXED POINT THEOREMS FOR A PAIR OF MAPPINGS IN b- DISLOCATED METRIC SPACE

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ABSTRACT. The purpose of this paper is to prove common fixed point theorems in complete b-dislocated metric space. We introduce $\Psi - G$ contraction where Ψ and G are the set of all continuous and non- decreasing functions to show the existence of unique common fixed point. Further we prove a fixed point theorem for extended $s - \alpha$ quasi-contraction.

1. INTRODUCTION

Fixed point theory (FPT) finds its applications in various fields like economics, engineering, physics, mathematics ,etc. The Banach contraction principle (BCP) is the base of research in metric space. In 1989, Bakhtin [6] and Czerwik [5], extended the BCP in metric space. Recently, we have seen a number of extensions of metric space. The dislocated metric space and b-dislocated metric space given by Hitzler and Seda [4] and Nawab Hussain et al. [3] respectively.

Several new applications have been suggested in [14] [12] [13] [15] [16] [24]. In 1976, Jungck [7] concluded the BCP using commuting mappings, entrenched the perception of mappings that are weakly commuting by Sessa [11] and the pair of mappings which are compatible is produced. Jungck and Rhoades [10] and Dhage [9] described if self-mappings pair commute at their coincidence points will be weakly compatible. Afterwards, this approach of

weak commutativity is weakened and used by Singh [22], Pathak [17], Mishra [21], Gairola and Whitfield [18], Pant [19], Tivari and Singh [20] and others. The purpose of the present research work is to introduce FPT for $s - \alpha$ quasi-contractions with the use of double self mappings in b-dislocated metric space which satisfies different set of constraints.We generalize some coincidence and fixed point theorems with concepts of weakly compatible pair of mappings, as well as by using Ψ -contractive conditions on b- dislocated metric spaces. The objective of this paper is to unify and generalize recent results in the setting dislocated and b -dislocated metric spaces using a class of continuous functions G_4 .

2. Preliminaries

Definition 2.1. *[1] Consider a non-empty set B with mapping $d_l : B \times B \to [0, \infty)$ is known as dislocated metric (d_l-metric) if the following constraints are satisfied for any $p, q, r \in B$:*

(1) *If $d_l(p, q) = 0$, then $p = q$;*
(2) *$d_l(p, q) = d_l(q, p)$;*
(3) *$d_l(p, q) \leqslant d_l(p, r) + d_l(r, q)$.*

The pair (B, d_l) is known as a dislocated metric space. But when $p = q, d_l(p, q)$ may not be 0.

Definition 2.2. *Consider $\{p_n\}$ is a sequence in d_l-metric space (B, d_l)*

(1) *if only, for $\epsilon > 0$, $\exists\ n_0$ belongs N s.t(such that) $\forall\ m, n \geqslant n_0$, we get $d_l(p_m, p_n) < \epsilon$ or $\lim_{n,m \to \infty} d_l(p_n, p_m) = 0$ then it is named as Cauchy sequence,*
(2) *it is convergent relative to d_l if $\exists$, $p \in B$ so that $d_l(p_n, p) \to 0$ as $n \to \infty$.*

By these circumstances, the limit of $\{p_n\}$ is p and $p_n \to p$.

Definition 2.3. *Consider a non empty set B with mapping*
$b_d : B \times B \to [0, \infty)$ is named as b-dislocated metric provided that the following constraints are satisfied for any $p, q, r \in B$ and $s \geqslant 1$:

(1) *If $b_d(p, q) = 0$, then $p = q$;*
(2) *$b_d(p, q) = b_d(q, p)$;*
(3) *$b_d(p, q) \leqslant s[b_d(p, r) + b_d(r, q)]$.*

The space (B, b_d) is known as b-dislocated metric space.

Definition 2.4. *[1] Consider (B, b_d) is a b_d-metric space, $\{p_n\}$ denotes sequence of points in B. Some point $p \in B$ is known as limit of $\{p_n\}$ provided $\lim_{n \to \infty} b_d(p_n, p) = 0$ then we assert $\{p_n\}$ is b_d-convergent to p and indicate it by $p_n \to p$ as $n \to \infty$.*

Definition 2.5. *[1] In a b_d-metric space (B, b_d) let a sequence $\{p_n\}$ is named as b_d-Cauchy sequence iff, for $\epsilon > 0$, $\exists$, n_0 belongs N s.t for all $n, m > n_0$, we are having $b_d(p_n, p_m) < \epsilon$ or $\lim_{n,m \to \infty} b_d(p_n, p_m) = 0$.*

Definition 2.6. *[2] Consider the self mappings pair (R, P) described on a metric space (B, d) is weakly - compatible. If the mappings commute at their coincidence points, i.e, $Rp = Pp$ for some $p \in B \Rightarrow RPp = PRp$.*

Definition 2.7. *[25] Consider a non-empty set B, R and P are two self-mappings then,*

   (i) *if $Rp = p$, then the point $p \in B$ is known as fixed point of R.*
   (ii) *if $Rp = Pp$, then the point $p \in B$ is known as coincidence point of R, P and $u = Rp = Pp$ is a coincidence point of R, P .*
   (iii) *if $Rp = Pp = p$, then the point $p \in B$ is known as common fixed point of R and P .*

We consider the set G_4 of all continuous functions $g : [0, \infty)^4 \to [0, \infty)$ with the properties:

   a) g is non-decreasing in respect to each variable.
   b) $g(t, t, t, t) \leqslant t, t \in [0, \infty)$.

Examples are as follows:

$g_1 : g_1(t_1, t_2, t_3, t_4) = max\{t_1, t_2, t_3, t_4\}$

$g_2 : g_2(t_1, t_2, t_3, t_4) = max\{t_1 + t_2, t_2 + t_3, t_1 + t_3, t_3 + t_4\}$

$g_3 : g_3(t_1, t_2, t_3, t_4) = [max\{t_1 t_2, t_2 t_3, t_1 t_3, t_3 t_4\}]^{\frac{1}{2}}$

$g_4 : g_4(t_1, t_2, t_3, t_4) = [max\{t_1^p, t_2^p, t_3^p, t_4^p\}]^{\frac{1}{p}}, p > 0$.

3. MAIN RESULTS

Consider Ψ which denotes the set of all continuous, non decreasing functions and $\Psi : [0, \infty) \to [0, \infty)$ s.t $\Psi(\alpha t) \geqslant 0$ iff $t = 0, \alpha > 0$.

Definition 3.1. *Let (B, b_d) be a complete b_d - dislocated metric space with parameter $s \geqslant 1$ and $R, P : B \to B$ are two self- mappings s.t $\forall \; p, q \in B$, $C \geqslant 2$ and $\Psi \in \Psi$ satisfy the following contractive condition*

$$\Psi(2s^2 b_d(Rp, Rq)) \leqslant C\Psi max[g\{b_d(Pp, Pq), b_d(Rp, Pp), b_d(Rq, Pq), \frac{b_d(Rp, Pq)b_d(Pp, Rq)}{2s}\},$$
$$g\{b_d(Rp, Rq), b_d(Pp, Rp), b_d(Rq, Pq), \frac{b_d(Rp, Pq)b_d(Rq, Pp)}{1 + b_d(Rp, Rq)}\}]$$

is known as $\Psi^\smile G$ contraction.

Theorem 3.1. *Let (B, b_d) be a complete b - dislocated metric space with parameter $s \geqslant 1$ and $R, P : B \to B$ are two self- mappings s.t $\forall \; p, q \in B$, $C \geqslant 2$ and $\Psi \in \Psi$ satisfying the following contractive condition:*

$$\Psi(2s^2 b_d(Rp, Rq)) \leqslant C\Psi max[g\{b_d(Pp, Pq), b_d(Rp, Pp), b_d(Rq, Pq), \frac{b_d(Rp, Pq)b_d(Pp, Rq)}{2s}\},$$
$$g\{b_d(Rp, Rq), b_d(Pp, Rp), b_d(Rq, Pq), \frac{b_d(Rp, Pq)b_d(Rq, Pp)}{1 + b_d(Rp, Rq)}\}]$$

where $g \in G_4, R(B) \subseteq P(B)$. Then mappings R and P have a unique common fixed point.

Proof. Consider p_0 be arbitrary element in B. As $R(B) \subseteq P(B)$.
Therefore, we can define a sequence,

$$\{Rp_0, Rp_1, Rp_2, ..., Rp_{2n}, Rp_{2n+1}, ...\}$$

s.t $Rp_{2n} = Pp_{2n+1}$ for n=0,1,2,3.... Now, to show that this sequence is a Cauchy sequence. Using the condition $p = p_{2n}$ and $q = p_{2n+1}$,we have

$$\Psi(2s^2 b_d(Rp_{2n}, Rp_{2n+1})) \leqslant C\Psi max[g\{b_d(Pp_{2n}, Pp_{2n+1}), b_d(Rp_{2n}, Pp_{2n}), b_d(Rp_{2n+1}, Pp_{2n+1}),$$
$$\frac{b_d(Rp_{2n}, Pp_{2n+1})b_d(Pp_{2n}, Rp_{2n+1})}{2s}\},$$
$$g\{b_d(Rp_{2n}, Rp_{2n+1}), b_d(Pp_{2n}, Rp_{2n}), b_d(Rp_{2n+1}, Pp_{2n+1}), \frac{b_d(Rp_{2n}, Pp_{2n+1})b_d(Rp_{2n+1}, Pp_{2n})}{1 + b_d(Rp_{2n}, Rp_{2n+1})}\}].$$

As $Rp_{2n} = Pp_{2n+1}$ for n=0,1,2,3.... we have,

$$\Psi(2s^2 b_d(Rp_{2n}, Rp_{2n+1})) \leqslant C\Psi max[g\{b_d(Rp_{2n-1}, Rp_{2n}), b_d(Rp_{2n}, Rp_{2n-1}), b_d(Rp_{2n+1}, Rp_{2n}),$$
$$\frac{b_d(Rp_{2n}, Rp_{2n})b_d(Rp_{2n-1}, Rp_{2n+1})}{2s}\},$$
$$g\{b_d(Rp_{2n}, Rp_{2n+1}), b_d(Rp_{2n-1}, Rp_{2n}), b_d(Rp_{2n+1}, Rp_{2n}), \frac{b_d(Rp_{2n}, Rp_{2n})b_d(Rp_{2n+1}, Rp_{2n-1})}{1 + b_d(Rp_{2n}, Rp_{2n+1})}\}]$$

$$\Psi(2s^2 b_d(Rp_{2n}, Rp_{2n+1})) \leqslant C\Psi max[g\{b_d(Rp_{2n-1}, Rp_{2n}), b_d(Rp_{2n}, Rp_{2n-1}), 0\},$$
$$g\{b_d(Rp_{2n}, Rp_{2n+1}), b_d(Rp_{2n-1}, Rp_{2n}), 0\}]$$

$$
\begin{aligned}
\Psi(2s^2 b_d(Rp_{2n}, Rp_{2n+1})) &\leqslant C\Psi[b_d(Rp_{2n-1}, Rp_{2n})] \\
b_d(Rp_{2n}, Rp_{2n+1}) &\leqslant \frac{C}{2s^2}[b_d(Rp_{2n-1}, Rp_{2n})] \\
b_d(Rp_{2n-1}, Rp_{2n}) &\leqslant \frac{C}{2s^2}[b_d(Rp_{2n-2}, Rp_{2n-1})] \\
b_d(Rp_{2n}, Rp_{2n+1}) &\leqslant kb_d(Rp_{2n-1}, Rp_{2n}) \leqslant \dots \leqslant k^{2n} b_d(Rp_0, Rp_1).
\end{aligned}
$$

Since, $0 \leqslant k < 1$ taking limit $n \to \infty$ we have: $b_d(Rp_{2n}, Rp_{2n+1}) \to 0$.
Further, we show $\{Rp_{2n}\}$ is b_d- Cauchy sequence.
Consider $m > 0$, $n > 0$ with $m > n$, by use of definition

$$b_d(Rp_{2n}, Rp_{2m}) \leqslant s\left[b_d(Rp_{2n}, Rp_{2n+1}) + b_d(Rp_{2n+1}, Rp_{2m})\right]$$

$$
\begin{aligned}
&\leqslant sb_d(Rp_{2n}, Rp_{2n+1}) + s^2 b_d(Rp_{2n+1}, Rp_{2n+2}) + s^3 b_d(Rp_{2n+2}, Rp_{2n+3}) \dots \\
&\leqslant sk^{2n} b_d(Rp_0, Rp_1) + s^2 k^{2n+1} b_d(Rp_0, Rp_1) + s^3 k^{2n+2} b_d(Rp_0, Rp_1) + \dots \\
&= sk^{2n} b_d(Rp_0, Rp_1)\left[1 + sk + (sk)^2 + (sk)^3 + \dots\right] \\
&\leqslant \frac{s}{1-sk} k^{2n} b_d(Rp_0, Rp_1).
\end{aligned}
$$

Taking limit for $(n, m) \to \infty$ we have $b_d(Rp_{2n}, Rp_{2m}) \to 0$ as $sk < 1$.
  Therefore, $\{Rp_{2n}\}$ is a b_d- Cauchy sequence in (B, b_d).
  Hence, $Rp_{2n} \to t$ and similarly $Pp_{2n} \to t$.
  By using definition of g we have

$$
\begin{aligned}
\Psi(2s^2 b_d(Rp_{2n}, Rp_{2n+1})) &\leqslant C\Psi max[\{b_d(Rp_{2n+1}, Rp_{2n}), b_d(Rp_{2n}, Rp_{2n+1}\} \\
\Psi(2s^2 b_d(Rp_{2n}, Rp_{2n+1})) &\leqslant C\Psi\{b_d(Rp_{2n+1}, Rp_{2n})\},
\end{aligned}
$$

we can write it as:

$$\Psi(2s^2 b_d(Pp_{2n+1}, Rp_{2n+1})) \leqslant C\Psi\{b_d(Rp_{2n+1}, Pp_{2n+1})\}.$$

As $p_{2n+1} \to t$ and R, P are continuous mappings so,we can have $Rp_{2n+1} \to Rt$
and $Pp_{2n+1} \to Pt$.

Now, by applying property of Ψ that is $\Psi(\alpha t) \geqslant 0$ iff $t = 0$, where $\alpha > 0$, $c \geqslant 2$ and $s \geqslant 1$, we have:

$$\Psi(2s^2 b_d(Pt, Rt)) \leqslant C\Psi\{b_d(Rt, Pt)\}$$
$$0 \leqslant \Psi\{(C - 2s^2)b_d(Rt, Pt)\} = 0\,,$$

because $\Psi(\alpha b_d(Pt, Rt)) \geqslant 0$, $(b_d(Pt, Rt)) = 0$, where $\alpha = (C - 2s^2)$. Therefore, we have $Pt = Rt$.

Put $p = t$, and $q = p_n$

$$\Psi(2s^2 b_d(Rt, Rp_n)) \leqslant C\Psi max[g\{b_d(Pt, Pp_n), b_d(Rt, Pt), b_d(Rp_n, Pp_n), \frac{b_d(Rt, Pp_n)b_d(Pt, Rp_n)}{2s}\},$$
$$g\{b_d(Rt, Rp_n), b_d(Pt, Rt), b_d(Rp_n, Pp_n), \frac{b_d(Rt, Pp_n)b_d(Rp_n, Pt)}{1 + b_d(Rt, Rp_n)}\}]\,.$$

Taking limit $n \to \infty$, we have

$$\Psi(2s^2 b_d(Rt, t)) \quad \leqslant \quad C\Psi max[g\{b_d(Pt, t), b_d(Rt, Pt), b_d(t, t), \frac{b_d(Rt, t)b_d(Pt, t)}{2s}\},$$
$$g\{b_d(Rt, t), b_d(Pt, Rt), b_d(t, t), \frac{b_d(Rt, t)b_d(t, Pt)}{1 + b_d(Rt, t)}\}]$$
$$\Psi(2s^2 b_d(Rt, t)) \quad \leqslant \quad C\Psi max[g\{b_d(Rt, t), b_d(Rt, Rt), b_d(t, t), \frac{b_d(Rt, t)b_d(Rt, t)}{2s}\},$$
$$g\{b_d(Rt, t), b_d(Rt, Rt), b_d(t, t), \frac{b_d(Rt, t)b_d(t, Rt)}{1 + b_d(Rt, t)}\}]\,.$$

Using definition of g

$$\Psi(2s^2 b_d(Rt, t)) \quad \leqslant \quad C\Psi max\{b_d(Rt, t), b_d(Rt, t)\}$$
$$\Psi(2s^2 b_d(Rt, t)) \quad \leqslant \quad C\Psi\{b_d(Rt, t)\}$$
$$0 \quad \leqslant \quad \Psi\{(C - 2s^2)b_d(Rt, t)\}$$
$$(b_d(Rt, t)) \quad = \quad 0$$
$$Rt \quad = \quad t = Pt\,.$$

Hence, common fixed point of (R, P) is $'t'$.

Uniqueness:

For (R, P), assume $t \neq t_0$ be two common fixed points.

Put $p = t, q = t_0$.

$$\Psi(2s^2 b_d(Rt, Rt_0)) \leqslant C\Psi max[g\{b_d(Pt, Pt_0), b_d(Rt, Pt), b_d(Rt_0, Pt_0), \frac{b_d(Rt, Pt_0)b_d(Pt, Rt_0)}{2s}\},$$
$$g\{b_d(Rt, Rt_0), b_d(Pt, Rt), b_d(Rt_0, Pt_0), \frac{b_d(Rt, Pt_0)b_d(Rt_0, Pt)}{1 + b_d(Rt, Rt_0)}\}]$$

$$\Psi(2s^2 b_d(t, t_0)) \leqslant C\Psi max[g\{b_d(t, t_0), b_d(t, t), b_d(t_0, t_0), \frac{b_d(t, t_0)b_d(t, t_0)}{2s}\},$$
$$g\{b_d(t, t_0), b_d(t, t), b_d(t_0, t_0), \frac{b_d(t, t_0)b_d(t_0, t)}{1 + b_d(t, t_0)}\}]$$
$$\Psi(2s^2 b_d(t, t_0)) \leqslant C\Psi max\{b_d(t, t_0)b_d(t, t_0)\}$$
$$\Psi(2s^2 b_d(t, t_0)) \leqslant C\Psi\{b_d(t, t_0)\}$$
$$0 \leqslant \Psi\{(C - 2s^2)b_d(t, t_0)\}$$
$$b_d(t, t_0) = 0$$
$$t = t_0.$$

Therefore fixed point is unique. $\qquad\qquad\qquad\square$

Example 1. *Let $(B, b_d) = [0, 1]$ is complete b-dislocated metric space on B, a function $g(t_1, t_2, t_3, t_4) = max\{t_1, t_2, t_3, t_4\}, \Psi(\alpha t) \geqslant 0$, where $s \geqslant 1$ is a parameter. Define a mapping which satisfies*

$$Rp = \begin{cases} \frac{1}{8}p & ; p \in [0, 1) \\ \frac{1}{6} & ; p = 1 \end{cases}$$

and

$$Pp = \begin{cases} \frac{1}{5}p & ; p \in [0, 1) \\ \frac{1}{3} & ; p = 1 \end{cases}$$

Clearly it is seen that $(B, b_d) = [0, 1]$ is complete b-dislocated metric space on B. Now, we satisfy following cases:

Case 1. *When $p, q \in [0, 1)]$ we have*

$$\Psi(2s^2 b_d(Rp, Rq)) \leqslant C\Psi max[g\{b_d(Pp, Pq), b_d(Rp, Pp), b_d(Rq, Pq), \frac{b_d(Rp, Pq)b_d(Pp, Rq)}{2s}\},$$
$$g\{b_d(Rp, Rq), b_d(Pp, Rp), b_d(Rq, Pq), \frac{b_d(Rp, Pq)b_d(Rq, Pp)}{1 + b_d(Rp, Rq)}\}],$$

and putting the value we have

$$\Psi(2s^2 b_d(\frac{p}{8}, \frac{q}{8})) \leqslant C\Psi max[g\{b_d(\frac{p}{5}, \frac{q}{5}), b_d(\frac{p}{8}, \frac{p}{5}), b_d(\frac{q}{8}, \frac{q}{5}), \frac{b_d(\frac{p}{8}, \frac{q}{5})b_d(\frac{p}{5}, \frac{q}{8})}{2s}\},$$
$$g\{b_d(\frac{p}{8}, \frac{q}{8}), b_d(\frac{p}{5}, \frac{p}{8}), b_d(\frac{q}{8}, \frac{q}{5}), \frac{b_d(\frac{p}{8}, \frac{q}{5})b_d(\frac{q}{8}, \frac{p}{5})}{1 + b_d(\frac{p}{8}, \frac{q}{8})}\}].$$

Using $p = \frac{1}{2}, q = \frac{1}{4}$
$\Psi(2s^2 b_d(\frac{p}{8}, \frac{q}{8})) \leqslant C\Psi max[\{b_d(\frac{p}{5}, \frac{q}{5}), b_d(\frac{p}{5}, \frac{p}{8})\}]$
$\Psi(2s^2 b_d(\frac{p}{8}, \frac{q}{8})) \leqslant C\Psi\{b_d(\frac{p}{5}, \frac{p}{8})\}$
and $s = 1$
$\Psi(0.03125) \leqslant C\Psi(0.06875)$
As Ψ is non-decreasing function and $C \geqslant 2$. Therefore $L.H.S \leqslant R.H.S$

Case 2. *When $q < p = 1$ we have:*

$$\Psi(2s^2 b_d(\frac{1}{6}, \frac{q}{8})) \leqslant C\Psi max[g\{b_d(\frac{1}{3}, \frac{q}{5}), b_d(\frac{1}{6}, \frac{1}{3}), b_d(\frac{q}{8}, \frac{q}{5}), \frac{b_d(\frac{1}{6}, \frac{q}{5})b_d(\frac{1}{3}, \frac{q}{8})}{2s}\},$$
$$g\{b_d(\frac{1}{6}, \frac{q}{8}), b_d(\frac{1}{3}, \frac{1}{6}), b_d(\frac{q}{8}, \frac{q}{5}), \frac{b_d(\frac{1}{6}, \frac{q}{5})b_d(\frac{q}{8}, \frac{1}{3})}{1 + b_d(\frac{1}{6}, \frac{q}{8})}\}]$$

using $p = 1, q = \frac{1}{2}$
$\Psi(2s^2 b_d(\frac{1}{6}, \frac{q}{8})) \leqslant C\Psi max\{b_d(\frac{1}{3}, \frac{q}{5}), b_d(\frac{1}{3}, \frac{1}{6})\}$
$\Psi(2s^2 b_d(\frac{1}{6}, \frac{q}{8})) \leqslant C\Psi\{b_d(\frac{1}{3}, \frac{q}{5})\}$
$\Psi(0.208) \leqslant C\Psi(0.167)$
As Ψ is non-decreasing function and $C \geqslant 2$. Therefore $L.H.S \leqslant R.H.S$

Case 3. *When $p < q = 1$ we have:*

$$\Psi(2s^2 b_d(\frac{p}{8}, \frac{1}{6})) \leqslant C\Psi max[g\{b_d(\frac{p}{5}, \frac{1}{3}), b_d(\frac{p}{8}, \frac{p}{5}), b_d(\frac{1}{6}, \frac{1}{3}), \frac{b_d(\frac{p}{8}, \frac{1}{3})b_d(\frac{p}{5}, \frac{1}{6})}{2s}\},$$
$$g\{b_d(\frac{p}{8}, \frac{1}{6}), b_d(\frac{p}{5}, \frac{p}{8}), b_d(\frac{1}{6}, \frac{1}{3}), \frac{b_d(\frac{p}{8}, \frac{1}{3})b_d(\frac{1}{6}, \frac{p}{5})}{1 + b_d(\frac{p}{8}, \frac{1}{6})}\}]$$

using $p = \frac{1}{2}, q = 1$
$\Psi(2s^2 b_d(\frac{p}{8}, \frac{1}{6})) \leqslant C\Psi max\{b_d(\frac{p}{8}, \frac{p}{5}), b_d(\frac{p}{5}, \frac{p}{8})\}$
$\Psi(2s^2 b_d(\frac{1}{6}, \frac{q}{8})) \leqslant C\Psi\{b_d(\frac{p}{5}, \frac{p}{8})\}$
$\Psi(-0.208) \leqslant C\Psi(0.038)$
As Ψ is non-decreasing function and $C \geqslant 2$. Therefore $L.H.S \leqslant R.H.S$

Case 4. *When $p = q = 1$ we have:*

$$\Psi(2s^2 b_d(\frac{1}{6}, \frac{1}{6})) \leqslant C\Psi max[g\{b_d(\frac{1}{3}, \frac{1}{3}), b_d(\frac{1}{6}, \frac{1}{3}), b_d(\frac{1}{6}, \frac{1}{3}), \frac{b_d(\frac{1}{6}, \frac{1}{3})b_d(\frac{1}{3}, \frac{1}{6})}{2s}\},$$

$$g\{b_d(\frac{1}{6}, \frac{1}{6}), b_d(\frac{1}{3}, \frac{1}{6}), b_d(\frac{1}{6}, \frac{1}{3}), \frac{b_d(\frac{1}{6}, \frac{1}{3})b_d(\frac{1}{6}, \frac{1}{3})}{1 + b_d(\frac{1}{6}, \frac{1}{6})}\}]$$

$\Psi(2s^2 b_d(\frac{1}{6}, \frac{1}{6})) \leqslant C\Psi max\{b_d(\frac{1}{6}, \frac{1}{3}), b_d(\frac{1}{3}, \frac{1}{6})\}$

$\Psi(2s^2 b_d(\frac{1}{6}, \frac{1}{6})) \leqslant C\Psi\{b_d(\frac{1}{3}, \frac{1}{6})\}$

$\Psi(0) \leqslant C\Psi(0.167)$

As Ψ is non-decreasing function and $C \geqslant 2$. Therefore $L.H.S \leqslant R.H.S$

Definition 3.2. *Let (B, b_d) is complete b-dislocated metric space and R, and P : $B \to B$ are self mappings which satisfy:*

$$s^2 b_d(Rp, Rq) \leqslant \alpha \max \{b_d(Rp, Rq), b_d(Rp, Pp), b_d(Rq, Pq), b_d(Rp, Pq), b_d(Rq, Pp)\}$$

for all p, q belongs B, $\alpha \in [0, \frac{1}{2}]$ and $s \geqslant 1$.
  Then R and P are called a $s - \alpha$ quasi-contraction.

Further, the existence of common fixed point for extended $s-\alpha$ quasi-contraction for two mappings on complete b-dislocated metric spaces is shown.

Theorem 3.2. *Consider the pair (R, P) of self mappings on a complete b-dislocated metric space (B, b_d) where $g \in G_4$, $\alpha \in [0, \frac{1}{2}]$ s.t*

$$s^2 b_d(Rp, Rq) \leqslant \alpha g \{b_d(Rp, Rq), b_d(Rp, Pp), b_d(Rq, Pq), b_d(Rp, Pq), b_d(Rq, Pp)\}$$

$R(B) \subseteq P(B)$ and R , P are weakly compatible. Then R and P have a unique common fixed point.

Proof. Consider p_0 be arbitrary element in B. Because $R(B) \subseteq P(B)$, we can define a sequence,

$$\{Rp_0, Rp_1, Rp_2, \ldots, Rp_{2n}, Rp_{2n+1}, \ldots\}$$

s.t $Rp_{2n} = Pp_{2n+1}$ for n is non-negative integers.
  Now, to show the sequence is a Cauchy sequence, using $p = p_{2n}$ and $q = p_{2n+1}$, we have:

$$s^2 b_d(Rp_{2n}, Rp_{2n+1}) \leqslant \alpha g\{b_d(Rp_{2n}, Rp_{2n+1}), b_d(Rp_{2n}, Pp_{2n}), b_d(Rp_{2n+1}, Pp_{2n+1}),$$

$$b_d(Rp_{2n}, Pp_{2n+1}), b_d(Rp_{2n+1}, Pp_{2n})\} .$$

As $Rp_{2n} = Pp_{2n+1}$ for n is non-negative integers we have:

$$s^2 b_d(Rp_{2n}, Rp_{2n+1}) \leqslant \alpha g\,\{b_d(Rp_{2n}, Rp_{2n+1}), b_d(Rp_{2n}, Pp_{2n}), b_d(Rp_{2n+1}, Rp_{2n}),\,,$$
$$b_d(Rp_{2n}, Rp_{2n})b_d(Rp_{2n+1}, Rp_{2n})\}\,,$$

$$s^2 b_d(Rp_{2n}, Rp_{2n+1}) \;\leqslant\; \alpha g\,\{b_d(Rp_{2n}, Rp_{2n+1}), b_d(Rp_{2n}, Pp_{2n}), 0, b_d(Rp_{2n+1}, Pp_{2n})\}$$
$$s^2 b_d(Rp_{2n}, Rp_{2n+1}) \;\leqslant\; \alpha g\,\{b_d(Rp_{2n}, Rp_{2n+1}), b_d(Rp_{2n}, Rp_{2n-1}), b_d(Rp_{2n+1}, Pp_{2n})\}$$
$$\leqslant\; \alpha\,\{b_d(Rp_{2n-1}, Rp_{2n})\}$$
$$b_d(Rp_{2n}, Rp_{2n+1}) \;\leqslant\; \frac{\alpha}{s^2} b_d(Rp_{2n-1}, Rp_{2n})\,.$$

Similarly,

$$b_d(Rp_{2n-1}, Rp_{2n}) \leqslant \frac{\alpha}{s^2} b_d(Rp_{2n-2}, Rp_{2n-1})\,.$$

Now we have , for all $n \geqslant 0$

$$b_d(Rp_{2n}, Rp_{2n+1}) \leqslant k b_d(Rp_{2n-1}, Rp_{2n}) \leqslant .... \leqslant k^{2n} b_d(Rp_0, Rp_1)\,,$$

$k = \frac{\alpha}{s^2}$; $0 \leqslant k < 1$.

Taking limit $n \to \infty$ we get $b_d(Rp_{2n}, Rp_{2n+1}) \to 0$.

Further, to prove $\{Rp_{2n}\}$ is b_d- Cauchy sequence, we consider $m > 0$, $n > 0$ with $m > n$, using definition

$$b_d(Rp_{2n}, Rp_{2m}) \leqslant s\,[b_d(Rp_{2n}, Rp_{2n+1}) + b_d(Rp_{2n+1}, Rp_{2m})]\,,$$

$$\leqslant\; s b_d(Rp_{2n}, Rp_{2n+1}) + s^2 b_d(Rp_{2n+1}, Rp_{2n+2}) + s^3 b_d(Rp_{2n+2}, Rp_{2n+3}) \ldots$$
$$\leqslant\; s k^{2n} b_d(Rp_0, Rp_1) + s^2 k^{2n+1} b_d(Rp_0, Rp_1) + s^3 k^{2n+2} b_d(Rp_0, Rp_1) + \ldots$$
$$=\; s k^{2n} b_d(Rp_0, Rp_1)\,\big[1 + sk + (sk)^2 + (sk)^3 + \ldots\big]$$
$$\leqslant\; \frac{s}{1 - sk} k^{2n} b_d(Rp_0, Rp_1)\,.$$

Taking limit for $(n, m) \to \infty$ we have $b_d(Rp_{2n}, Rp_{2m}) \to 0$ as $sk < 1$.

Therefore, $\{Rp_{2n}\}$ is a b_d- Cauchy sequence in (B, b_d).

Since, $\lim_{n \to \infty} Rp_{2n} = \lim_{n \to \infty} Pp_{2n+1} = h \in P(B)$ therefore, $\exists\, l \in B$ s.t $P(l) = h$. We claim that $Pl = Rl$. If not then with $p = l$, $q = p_{2n}$, we have

$$s^2 b_d(Rl, Rp_{2n}) \leqslant \alpha g\,\{b_d(Rl, Rp_{2n}), b_d(Rl, Pl), b_d(Rp_{2n}, Pp_{2n}), b_d(Rl, Pp_{2n}), b_d(Rp_{2n}, Pl)\}\,.$$

Taking $\lim_{n\to\infty}$ we get,

$$s^2 b_d(Rl, h) \leqslant \alpha g \{b_d(Rl, h), b_d(Rl, Pl), b_d(h, h), b_d(Rl, h), b_d(h, Pl)\}$$

$$s^2 b_d(Rl, Pl) \leqslant \alpha g \{b_d(Rl, h), b_d(Rl, Pl), 0, b_d(h, h), b_d(h, h)\}$$

$$s^2 b_d(Rl, Pl) \leqslant \alpha g \{b_d(Rl, Pl), b_d(Rl, Pl), 0, b_d(h, h), b_d(h, h)\}$$

$$s^2 b_d(Rl, Pl) \leqslant \alpha \{b_d(Rl, Pl)\} .$$

$$b_d(Rl, Pl) \leqslant \frac{\alpha}{s^2} \{b_d(Rl, Pl)\} < b_d(Rl, Pl) ,$$

which is a contradiction, since $\dfrac{\alpha}{s^2} < 1$.

Hence, $Rl = Pl$, and therefore coincidence point of (R, P) is $'l'$. We are given with weakly compatible pair (R, P). So, $Ph = PRl = RPl = Rh$.

Next, we prove h is a common fixed point. We claim that, $Rh = h$. Again using $p = h, y = l$

$$s^2 b_d(Rh, Rl) \leqslant \alpha g \{b_d(Rh, Rl), b_d(Rh, Ph), b_d(Rl, Pl), b_d(Rh, Pl), b_d(Rl, Ph)\}$$

$$s^2 b_d(Rh, h) \leqslant \alpha g \{b_d(Rh, h), b_d(Rh, Ph), b_d(h, h), b_d(Rh, h), b_d(h, Ph)\}$$

$$s^2 b_d(Rh, h) \leqslant \alpha g \{b_d(Rh, h), b_d(Rh, Rh), b_d(h, h), b_d(Rh, h), b_d(h, Rh)\}$$

$$s^2 b_d(Rh, h) \leqslant \alpha g \{b_d(Rh, h)\} .$$

$$b_d(Rh, h) \leqslant \frac{\alpha}{s^2} \{b_d(Rh, h)\} ,$$

which is not possible, since $\frac{\alpha}{s^2} < 1$.

Hence $Rh = h$ or $Ph = h$. Therefore h is common fixed point of (R, P). $\square$

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