Advances in Mathematics: Scientific Journal **9** (2020), no.7, 4761–4767
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.7.42

SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS BY MODIFIED ADOMIAN DECOMPOSITION METHOD

NUHA MOHAMMED DABWAN AND YAHYA QAID HASAN[1]

ABSTRACT. In this study, we have solved the fractional differential equation by using the Modified Adomian Decomposition Method (MADM). The approximate analytical solution of this equation is obtained.

1. INTRODUCTION

Fractional differential equations are very important in many fields like fluid mechanics, biology, physics, engineering, electrochemistry of corrosion, viscoelastic, and electrical networks [4].

The Adomian decomposition method [1-3,5] is one of the most frequently used for computing solutions of linear and non-linear ordinary, partial, fractional differential equations, where it introduced by mathematician George Adomian in 1980s. For the case $\alpha = 1$, Hasan has obtained the approximate analytical solution to this equation by using Adomian decomposition method [8]. The goal of this study to introduce a new differential operator to study singular and nonsingular fractional differential equations.

[1] *corresponding author*

2010 *Mathematics Subject Classification.* 34A08.

Key words and phrases. Fractional differential equation, Adomian decomposition method.

2. Adomian Method

In this study, we consider the fractional differential equation:

$$D_x^\alpha y + p(x^\alpha)y = f(x) - g(x,y), \tag{2.1}$$

subject to condition

$$y(0) = A,$$

where $0 < \alpha \leq 1$,
we write equation (2.1) in the standard operator form

$$L_\alpha y = f(x) - g(x,y), \tag{2.2}$$

where

$$L_\alpha(.) = e^{\int -p(x^\alpha)dx^\alpha} \frac{d^\alpha}{dx^\alpha} e^{\int p(x^\alpha)dx^\alpha}(.),$$

and the inverse fractional operator L_α^{-1} is given by

$$L_\alpha^{-1}(.) = e^{\int -p(x^\alpha)dx^\alpha} \int_0^x e^{\int p(x^\alpha)dx^\alpha}(.)dx^\alpha. \tag{2.3}$$

By applying equation (2.3) on equation (2.2), we get

$$y(x) = \delta(x) + L_\alpha^{-1} f(x) - L_\alpha^{-1} g(x,y). \tag{2.4}$$

The general solution of the given equation is decomposed into the sum

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \tag{2.5}$$

The non-linear part can be decomposed into the infinite polynomial series obtained by

$$g(x,y) = \sum_{n=0}^{\infty} A_n, \tag{2.6}$$

where the elements $y_n(x)$ of the solution $y(x)$ will be determined repeatable. Specific algorithms were seen [6,7] to formulate Adomian polynomials. The following algorithm:

$$
\begin{aligned}
A_0 &= G(y_0),\\
A_1 &= y_1 G'(y_0),\\
A_2 &= y_2 G'(y_0) + \frac{1}{2!} y_1^2 G''(y_0),\\
A_3 &= y_3 G'(y_0) + y_1 y_2 G''(y_0) + \frac{1}{3!} y_1^3 G'''(y_0),\\
&\dots
\end{aligned}
\tag{2.7}
$$

can be used to build Adomian polynomials, when $G(y)$ is any function. From (2.4),(2.5) and (2.6) we have

$$\sum_{n=0}^{\infty} y_n(x) = \delta(x) + L_\alpha^{-1} f(x) - + L_\alpha^{-1} \sum_{n=0}^{\infty} A_n.$$

The component $y(x)$ can be given by using Adomian decomposition method as follows

$$
\begin{aligned}
y_0 &= \delta(x) + L_\alpha^{-1} f(x),\\
y_{(n+1)} &= -L_\alpha^{-1} A_n, \ n \geq 0,
\end{aligned}
$$

thus

$$
\begin{aligned}
y_0 &= \delta(x) + L_\alpha^{-1} f(x),\\
y_1 &= L_\alpha^{-1} A_0,\\
y_2 &= L_\alpha^{-1} A_1,\\
y_3 &= L_\alpha^{-1} A_2,\\
&\dots
\end{aligned}
\tag{2.8}
$$

Using the equations (2.7) and (2.8) we can determine the components $y_n(x)$, and hence the series solution of $y(x)$ in (2.5) can be immediately obtained. For numerical purposes, the n-term approximate

$$\zeta_n = \sum_{k=0}^{n-1} y_k,$$

can be used to approximate the exact solution.

3. APPLICATIONS

In this section, some examples are given to demonstrate the applicability and accuracy of our method.

Example 1. *Consider the linear fractional differential equation:*

$$D_x^\alpha y + \frac{\alpha}{x^\alpha} y = \alpha \cos x^\alpha + \frac{\alpha}{x^\alpha} \sin x^\alpha, \tag{3.1}$$

$$y(0) = 0.$$

The exact solution is $y(x) = \sin x^\alpha$. *We put*

$$L_\alpha(.) = \frac{1}{x^\alpha} \frac{d^\alpha}{dx^\alpha} x^\alpha(.).$$

So

$$L_\alpha^{-1}(.) = \frac{1}{x^\alpha} \int_0^x x^\alpha(.) dx^\alpha.$$

In an operator form equation (3.1) becomes

$$L_\alpha y = \alpha \cos x^\alpha + \frac{\alpha}{x^\alpha} \sin x^\alpha. \tag{3.2}$$

By applying L_α^{-1} *to both side of (3.2) we have*

$$y(x) = \sin x^\alpha.$$

Example 2. *Consider the nonlinear fractional differential equation:*

$$D_x^\alpha y + 3\alpha x^{2\alpha} y = \alpha e^{x^\alpha} + 3\alpha y (\ln y)^2, \tag{3.3}$$

$$y(0) = 1.$$

The exact solution is $y(x) = e^{x^\alpha}$. *We put*

$$L_\alpha(.) = e^{-x^{3\alpha}} \frac{d^\alpha}{dx^\alpha} e^{x^{3\alpha}}(.).$$

So

$$L_\alpha^{-1}(.) = e^{-x^{3\alpha}} \int_0^x e^{x^{3\alpha}}(.) dx^\alpha.$$

In an operator form equation (3.3) becomes

$$L_\alpha y = \alpha e^{x^\alpha} + 3\alpha y (\ln y)^2. \tag{3.4}$$

By applying L_α^{-1} *to both side of (3.4) we have*

$$y(x) = e^{-x^{3\alpha}} + e^{-x^{3\alpha}} \int_0^x e^{x^{3\alpha} + x^\alpha}(.) dx^\alpha + 3\alpha L_\alpha^{-1} y (\ln y)^2.$$

$$y_0 = e^{-x^{3\alpha}} + e^{-x^{3\alpha}} \int_0^x e^{x^{3\alpha}+x^{\alpha}} (.) dx^{\alpha}.$$

$$y_{n+1} = 3\alpha L_{\alpha}^{-1} A_n, \quad n \geq 0.$$

Then

$$y_0 = 1 + x^{\alpha} + \frac{x^{2\alpha}}{2} - \frac{5\,x^{3\alpha}}{6} - \frac{17\,x^{4\alpha}}{24} - \frac{7\,x^{5\alpha}}{24} + \frac{301\,x^{6\alpha}}{720} + \frac{1531\,x^{7\alpha}}{5040} + \frac{4411\,x^{8\alpha}}{40320} + \dots,$$

$$y_1 = x^{3\alpha} + \frac{3\,x^{4\alpha}}{4} - \frac{9\,x^{5\alpha}}{10} - \frac{5\,x^{6\alpha}}{3} - \frac{127\,x^{7\alpha}}{280} + \frac{353\,x^{8\alpha}}{320} + \dots,$$

$$y_2 = \frac{6\,x^{5\alpha}}{5} + \frac{5\,x^{6\alpha}}{4} - \frac{183\,x^{7\alpha}}{140} - \frac{253\,x^{8\alpha}}{80} + \dots,$$

$$y_3 = \frac{51\,x^{7\alpha}}{35} + \frac{39\,x^{8\alpha}}{20} - \frac{1027\,x^{9\alpha}}{560} - \frac{51\,x^{10\alpha}}{35} + \dots,$$

therefore

$y(x) = y_0 + y_1 + y_2 + y_3 =$

$$1 + x^{\alpha} + \frac{x^{2\alpha}}{2} + \frac{x^{3\alpha}}{6} + \frac{x^{4\alpha}}{24} + \frac{x^{5\alpha}}{120} + \frac{x^{6\alpha}}{720} + \frac{x^{7\alpha}}{5040} + \frac{x^{8\alpha}}{40320} - \frac{1027\,x^{9\alpha}}{560} - \frac{51\,x^{10\alpha}}{35} + \dots.$$

TABLE 1. Approximate Solution of Example 2 for different values of α and absolute error at $\alpha = 1$

	Approximate solutions by ADM	*Exact*	*Error*			
x	$\alpha = 0.99$	$\alpha = 0.98$	$\alpha = 0.97$	$\alpha = 1$	$\alpha = 1$	$y_{Exact} - y_{MADM}$
0.0	*1.000*	*1.000*	*1.000*	*1.000*	*1.000*	*0.000*
0.1	*1.10775*	*1.11039*	*1.1131*	*1.10517*	*1.10517*	*0.000*
0.3	*1.35472*	*1.35971*	*1.36477*	*1.34981*	*1.34986*	*0.00005*
0.5	*1.64913*	*1.65458*	*1.6600*	*1.64372*	*1.64872*	*0.00500*
0.7	*1.89972*	*1.90075*	*1.90168*	*1.89859*	*2.01375*	*0.11516*
0.9	*1.23121*	*1.22128*	*1.21123*	*1.24103*	*2.4596*	*1.21857*

Example 3. *Consider the non-linear fractional differential equation:*

(3.5) $$D_x^{\alpha} y + \alpha x^{\alpha} y = (2\alpha + \alpha x^{2\alpha} + x^{3\alpha}) x^{\alpha} - y^2,$$

$$y(0) = 0.$$

The exact solution is $y(x) = x^{2\alpha}$. *We put*

$$L_\alpha(.) = e^{-\frac{x^{2\alpha}}{2}} \frac{d^\alpha}{dx^\alpha} e^{\frac{x^{2\alpha}}{2}} (.).$$

So

$$L_\alpha^{-1}(.) = e^{-\frac{x^{2\alpha}}{2}} \int_0^x e^{\frac{x^{2\alpha}}{2}} (.) dx^\alpha.$$

In an operator form equation (3.5) becomes

$$L_\alpha y = (2\alpha + \alpha x^{2\alpha} + x^{3\alpha}) x^\alpha - y^2. \tag{3.6}$$

By applying L_α^{-1} *to both side of (3.6) we have*

$$y(x) = L_\alpha^{-1}((2\alpha + \alpha x^{2\alpha} + x^{3\alpha}) x^\alpha) - L_\alpha^{-1} y^2.$$

$$y_0 = x^{2\,\alpha} + \frac{x^{5\,\alpha}}{5\,\alpha} - \frac{x^{7\,\alpha}}{35\,\alpha} + \frac{x^{9\,\alpha}}{315\,\alpha} + \dots,$$

$$y_{n+1} = -L_\alpha^{-1} A_n, \quad n \geq 0.$$

Then

$$y_0 = x^{2\,\alpha} + \frac{x^{5\,\alpha}}{5\,\alpha} - \frac{x^{7\,\alpha}}{35\,\alpha} + \frac{x^{9\,\alpha}}{315\,\alpha} + \dots,$$

$$y_1 = \frac{-x^{8\,\alpha}}{20\,\alpha^2} + \frac{3\,x^{10\,\alpha}}{280\,\alpha^2} - \frac{x^{5\,\alpha}}{5\,\alpha} + \frac{x^{7\,\alpha}}{35\,\alpha} - \frac{x^{9\,\alpha}}{315\,\alpha} + \dots,$$

$$y_2 = \frac{x^{8\,\alpha}}{20\,\alpha^2} - \frac{3\,x^{10\,\alpha}}{280\,\alpha^2} + \dots,$$

therefore

$$y(x) = y_0 + y_1 + y_2 = x^{2\,\alpha}.$$

4. Conclusions

The ADM is a powerful tool in applied mathematics and engineering and have been applied for solving linear and non-linear differential equation. In this study, the application of MADM is investigated to obtain the exact solution of linear and non-linear fractional differential equations. Solving some examples show that the MADM is efficient and easy techniques for obtaining analytic solution of linear and non-linear fractional differential equations.

REFERENCES

[1] G. ADOMIAN, R. RACH: *itInversion of nonlinear stochastic operators,* J.Math. Anal. Appl., **91** (1983), 39-46.

[2] G. ADOMIAN: *A review of the decomposition in applied mathematics,* Math. ÙŘÙŘAnal. Appl., **1** (1990), 501-544.

[3] G. ADOMIAN: *A review of the decomposition method and some recent results for nonlinear equations,* Math. Comput. Appl., **1** (1990), 17-43.

[4] G. SAMKO, A. A. KILBAS, O. I. MARICHEV: *Fractional Integrals and Derivatives: Theory and Applications,* Gordon and Breach, Yverdon, 1993.

[5] G. ADOMIAN: *Solving frontier problem of physics: the decomposition method*, Kluwer academic publishers London, 1994.

[6] A. M. WAZAZ: *A First Course in Integral Equation,* World Scientific, Singapore, 1997.

[7] A. M. WAZWAZ: *A new algorithm for calculating Adomian polynomials for nonlinear operators,* Appl. Math. Comput., **111** (2000), 53-69.

[8] Y. Q. HASAN: *Solving first order ordinary differential equation by modified Adomian decomposition method,* (AITS), **1** (4) (2012), 86-89.

DEPARTMENT OF MATHEMATICS
SHEBA REGION UNIVERSITY
YEMEN
E-mail address: nuhamdm@gmail.com

DEPARTMENT OF MATHEMATICS
SHEBA REGION UNIVERSITY
YEMEN
E-mail address: yahya217@yahoo.com