SOME INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR PRODUCT HARMONIC CONVEX FUNCTION

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ABSTRACT. The objective of this paper is to establish the integral inequalities of Hermite-Hadamard type for product of harmonic convex function of single variable and several variables, product of exponential convex function and product of exponential harmonic function.

1. INTRODUCTION

Convex function and convex set have been studied intensively in mathematical engineering, management science and optimization theory. The classical convexity has been extended and generalized in different direction such as invexity by which the variational inequality problems studied on Banach spaces,Hilbert spaces etc. by B.Kodamasingh et al see [4–6]. Hermite-Hadamard type inequality was studied under the various convex functions. Further, it is extended on harmonic convex function by Imdat Iscan see [1]. Many studies have shown that the theory of harmonic convex function is related with the theory of inequalities.By using the harmonically convex function the bounds of the integral of convex function can be easily obtained. Inequalities play an Important role in many branches of sciences. A great number of studies have derived that the concept of Harmonic convex function is closely related to the concept of Inequalities. One of the most used inequalities for convex function named Hermite Hadamard integral inequality is

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furthur studied by many researchers like Imdat Iscan see [1], Dragomir see [2], Noor see [3], Mishra see [7], Noor see [8], Anderson see [9], Hap see [10], Baloch see [11].

Let $\mathcal{L}[c, d]$ be the set of all integrable functions in the interval $[c, d]$. Consider a subset K of R with R not equal to 0 is a harmonic convex set, $f : K \to R$ be a harmonically convex function and $c, d \in K$ with $c < d$. If $g \in \mathcal{L}[c, d]$, then the following inequality is called Hermite-Hadamard-type-Inequality.

$$(1.1) \qquad g\left(\frac{2cd}{c+d}\right) \leq \frac{cd}{d-c} \int_c^d \frac{g(x)}{x^2} dx \leq \frac{g(c) + g(d)}{2}.$$

2. PRELIMINARIES AND NOTATIONS

Definition 2.1. *Consider a subset $\mathcal{K}$ of R with R not equal to 0 and $g : \mathcal{K} \to R$ be any map then definitions are described as follows.*

(1) *[1] The set $\mathcal{K}$ is said to be a harmonically convex set if $\dfrac{xy}{tx + (1-t)y} \in \mathcal{K}$ for all $x, y \in \mathcal{K}$ and $t \in [0, 1]$.*

(2) *[1] The mapping g is said to be a harmonically convex function on the harmonically convex set $\mathcal{K}$ if*

$$g\left(\frac{xy}{tx + (1-t)y}\right) \leq tg(y) + (1-t)g(x)$$

for all $x, y \in \mathcal{K}$ and $t \in [0, 1]$.

Definition 2.2. *Consider a subset $\mathcal{K}$ of R with R not equal to 0 and $G : \mathcal{K} \to R$ be any map then definitions are described as follows.*

(1) *[3] A positive function G is said to be exponentially convex function if*

$$e^{G((1-t)a+tb)} \leq (1-t)e^{G(a)} + te^{G(b)} \ \forall a, b \in \mathcal{K}; t \in [0, 1]$$

(2) *[3] A positive function G is said to be exponentially harmonic convex if*

$$e^{G\left(\frac{ab}{ta+(1-t)b}\right)} \leq e^{tG(b)+(1-t)G(a)}$$

for all a,b$\in \mathcal{K}$.

The product of the function $f(x)$ and the function $g(x)$ is denoted by $f(x) * g(x)$ or $f * g$ throughout the paper.

3. MAIN RESULTS

In this section we have extended the result of Hermite-Hadamard type inequality in (1.1) under the product harmonic function.

Theorem 3.1. *Let* $g, h : K \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *be harmonic functions and* $g, h \in \mathcal{L}[c, d]$, *with* $c < d$. *If* $g(x) * h(x) \in \mathcal{L}[c, d]$, *then for all* $\alpha \geq 1$

$$\frac{cd}{d-c} \int_c^d \frac{g(x) * h(x)}{x^2} dx \leq \frac{\alpha}{3} M(c, d) + \frac{\alpha}{6} N(c, d),$$

where $M(c, d) = g(c) h(c) + g(d) h(d)$ *and* $N(c, d) = g(c) h(d) + g(d) h(c)$.
i.e Integral of the product harmonic function is dominated by the Linear Combination of $M(c, d)$ *and* $N(c, d)$.
In particular integral of the product function $g(x) * h(x)$ *is dominated by the convex combination of M(c,d) and N(c,d) for* $\alpha = 2$, *and least upper bound of the integral when* $\alpha = 1$

Proof. g is a harmonic function ,

$$g\left(\frac{cd}{sc + (1-s)d}\right) \leq sg(d) + (1-s)g(c)$$

for every c,d $\in$ K and $s \in [0, 1]$.
h is a harmonic function ,

$$h\left(\frac{cd}{sc + (1-s)d}\right) \leq sh(d) + (1-s)h(c)$$

for every c,d $\in$ K and $s \in [0, 1]$.
Multiplying the above two inequalities, and for any $\alpha \geq 1$, we have:

$$\left[g\left(\frac{cd}{sc + (1-s)d}\right)\right]\left[h\left(\frac{cd}{sc + (1-s)d}\right)\right]$$

$$\leq \alpha \left[s^2 g(d)h(d) + (1-s)^2 g(c)h(c) + s(1-s)g(c)h(d) + s(1-s)g(d)h(c)\right].$$

Put $x = \dfrac{cd}{sc + (1-s)d}$ and $dx = \dfrac{cd(d-c)}{(sc + (1-s)d)^2}$, and integrating w.r.t s over $[0, 1]$,

$$\int_0^1 g\left(\frac{cd}{sc + (1-s)d}\right) h\left(\frac{cd}{sc + (1-s)d}\right) ds = \frac{cd}{d-c} \int_c^d \frac{g(x) * h(x)}{x^2} dx$$

$$\leq \frac{\alpha}{3} [g(c)h(c) + g(d)h(d)] + \frac{\alpha}{6} [g(c)h(d) + g(d)h(c)]$$

$$= \frac{\alpha}{3} M(c, d) + \frac{\alpha}{6} N(c, d).$$

For convex combination

$$\frac{\alpha}{3} + \frac{\alpha}{6} = 1 \,,$$

gives $\alpha = 2$, when $\alpha = 1$, the integral has the least upper bound is obvious. □

Example 1. *Let $g(x) = x$ and $h(x) = \dfrac{(x-1)^2 + 1}{x}$ be harmonic convex functions on $(0, \infty)$. Consider the interval $[2, 5]$ and $g(x).h(x) = (x-1)^2 + 1 \in \mathcal{L}(2,5)$. Now*

$$\frac{g(x)h(x)}{x^2} = \frac{x^2 - 2x + 2}{x^2}$$

$$\frac{cd}{d-c} \int_c^d \frac{g(x)h(x)}{x^2} dx = \frac{10}{3} \int_2^5 \frac{x^2 - 2x + 2}{x^2} dx = 5.89$$

and for $\alpha = 1$, we have

$$\frac{1}{3} M(2,5) + \frac{1}{6} N(2,5) = 8.296 \,.$$

The inequality holds.

Theorem 3.2. *Let $g, h : K \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ are linear harmonic functions defined on $[c, d]$ and $g, h \in \mathcal{L}[c, d]$, with $c < d$. If the product $g(x) * h(x) \in \mathcal{L}[c, d]$, then*

$$0 \leq g\left(\frac{2cd}{c+d}\right) * h\left(\frac{2cd}{c+d}\right) \leq \frac{cd}{2(d-c)} \int_c^d \frac{g(x) * h(x)}{x^2} dx + \frac{M(c,d) + 2N(c,d)}{12}$$

where
$M(c,d) = g(c)h(c) + g(d)h(d)$ *and* $N(c,d) = g(c)h(d) + g(d)h(c)$.

Proof. Since $g(x)$ and $h(x)$ are linear harmonic convex functions and $harmonicmean(H.M) \leq arithmeticmean(A.M)$ then

$$g\left(\frac{2xy}{x+y}\right) \leq \frac{g(y) + g(x)}{2}$$

and

$$h\left(\frac{2xy}{x+y}\right) \leq \frac{h(y) + h(x)}{2} \,.$$

Taking

$$x = \frac{cd}{sd + (1-s)c}, \quad y = \frac{cd}{sc + (1-s)d}$$

and multiplying the two inequalities, we have:

$$g\left(\frac{2cd}{c+d}\right) * h\left(\frac{2cd}{c+d}\right) \leq \left[\frac{g\left(\frac{cd}{sc+(1-s)d}\right) + g\left(\frac{cd}{sd+(1-s)c}\right)}{2}\right] \left[\frac{h\left(\frac{cd}{sc+(1-s)d}\right) + h\left(\frac{cd}{sd+(1-s)c}\right)}{2}\right]$$

$$\leq \frac{1}{4}\left[g\left(\frac{cd}{sc+(1-s)d}\right) * h\left(\frac{cd}{sc+(1-s)d}\right) + g\left(\frac{cd}{sd+(1-s)c}\right) * h\left(\frac{cd}{sd+(1-s)c}\right)\right]$$

$$+\frac{1}{4}\left[[sg(d)+(1-s)f(c)] \times [sh(c)+(1-s)h(d)] + [sg(c)+(1-s)f(d)] \times [sh(d)+(1-s)h(c)]\right]$$

$$= \frac{1}{4}\left[g\left(\frac{cd}{sc+(1-s)d}\right) * h\left(\frac{cd}{sc+(1-s)d}\right) + g\left(\frac{cd}{sd+(1-s)c}\right) * h\left(\frac{cd}{sd+(1-s)c}\right)\right]$$

$$+\frac{1}{4}\left[2s(1-s)M(c,d) + \left(s^2+(1-s)^2\right)N(c,d)\right].$$

Integrating w.r.t s over $[0,1]$

$$g\left(\frac{2cd}{c+d}\right) * h\left(\frac{2cd}{c+d}\right) \leq \frac{1}{4}\left[\int_0^1 g\left(\frac{cd}{sc+(1-s)d}\right) * h\left(\frac{cd}{sc+(1-s)d}\right) ds\right.$$

$$+\left.\int_0^1 g\left(\frac{cd}{sd+(1-s)c}\right) * h\left(\frac{cd}{sd+(1-s)c}\right) ds\right]$$

$$+\frac{1}{4}\left[M(c,d)\int_0^1 2s(1-s)ds + N(c,d)\int_0^1 \left(s^2+(1-s)^2\right)ds\right],$$

i.e,

(3.1) $\quad g\left(\dfrac{2cd}{c+d}\right) * h\left(\dfrac{2cd}{c+d}\right) \leq \dfrac{cd}{2(d-c)}\displaystyle\int_c^d \dfrac{g(x)*h(x)}{x^2}dx + \dfrac{M(c,d)+2N(c,d)}{12}.$

Again,

$$0 \leq g\left(\frac{2xy}{x+y}\right) \text{ and } 0 \leq h\left(\frac{2xy}{x+y}\right).$$

Taking

$$x = \frac{cd}{sd+(1-s)c} \text{ and } y = \frac{cd}{sc+(1-s)d}$$

and multiplying the two inequalities, we have:

(3.2) $$0 \leq g\left(\frac{2cd}{c+d}\right) * h\left(\frac{2cd}{c+d}\right)$$

Now combining equation (3.1) and (3.2), we get

$$0 \leq g\left(\frac{2cd}{c+d}\right) * h\left(\frac{2cd}{c+d}\right) \leq \frac{cd}{2(d-c)}\int_c^d \frac{g(x)*h(x)}{x^2}dx + \frac{M(c,d)+2N(c,d)}{12}.$$

$$\square$$

Corollary 3.1. *Let $f(u)$ and $\bigtriangledown^2 g(u)$ are harmonic convex functions of more than one variable under the assumption of Theorem 3.2. If the product $\bigtriangledown f(u) * \bigtriangledown g(u)$*

$\in \mathcal{L}[a, b]$, *then*

$$2 \bigtriangledown f\left(\frac{2ab}{a+b}\right) \bigtriangledown g\left(\frac{2ab}{a+b}\right) - \frac{M(a,b) + 2N(a,b)}{12} \quad \leq \quad \frac{ab}{(b-a)} \int_a^b \frac{\bigtriangledown f(u) * \bigtriangledown g(u)}{u^2} du$$
$$\leq \quad [K_1 - K_2]$$

where $K_1 = f(u(1)). \bigtriangledown g(u(1)), K_2 = f(u(0)). \bigtriangledown g(u(0))$.

Proof. Consider

$$0 \leq \int_0^1 f(u). \bigtriangledown^2 g(u) dt \, .$$

Integrating by parts,

$$0 \quad \leq \quad [f(u(1)). \bigtriangledown g(u(1)) - f(u(0)). \bigtriangledown g(u(0))] - \int_0^1 \bigtriangledown f(u) * \bigtriangledown g(u) dt$$

$$\implies \quad \int_0^1 \bigtriangledown f(u) * \bigtriangledown g(u) dt \leq [f(u(1)). \bigtriangledown g(u(1)) - f(u(0)). \bigtriangledown g(u(0))] \, ,$$

for $u = \dfrac{ab}{ta + (1-t)b}$, $dt = \dfrac{ab}{u^2(a-b)} du$ and let $K_1 = f(u(1)). \bigtriangledown g(u(1))$, $K_2 = f(u(0)). \bigtriangledown g(u(0))$ we have:

$$(3.3) \qquad\qquad \frac{ab}{b-a} \int_a^b \frac{\bigtriangledown f(u) * \bigtriangledown g(u)}{u^2} du \leq [K_1 - K_2] \, .$$

f and g are harmonic convex function , $\bigtriangledown f$ and $\bigtriangledown g$ are harmonic convex function, using Theorem 3.2. Further,

(3.4)
$$2 \bigtriangledown f\left(\frac{2ab}{a+b}\right) \bigtriangledown g\left(\frac{2ab}{a+b}\right) - \frac{M(a,b) + 2N(a,b)}{12} \leq \frac{ab}{(b-a)} \int_a^b \frac{\bigtriangledown f(u) \bigtriangledown g(u)}{u^2} du$$

and from (3.3) and (3.4), we have

$$2 \bigtriangledown f\left(\frac{2ab}{a+b}\right) \bigtriangledown g\left(\frac{2ab}{a+b}\right) - \frac{M(a,b) + 2N(a,b)}{12} \quad \leq \quad \frac{ab}{(b-a)} \int_a^b \frac{\bigtriangledown f(u) \bigtriangledown g(u)}{u^2} du$$
$$\leq \quad [K_1 - K_2] \, .$$

$\square$

4. PRODUCT OF EXPONENTIAL CONVEX FUNCTION

In this section we have discussed Hermite Hadamard integral Inequality on product of exponentially convex function

Theorem 4.1. *Let the mapping $F, G : \mathcal{K} \subset \mathbb{R} \setminus \{0\} \to \mathbb{R} \; c, d \in \mathcal{K}, with c < d$ be exponential convex functions defined on $[c, d]$ such that $F, G \in L[c, d]$. If the product is exponentially convex function and $F * G \in L[c, d]$ then,*

$$\frac{1}{d-c} \int_c^d e^{F(x)+G(x)} dx \leq \frac{1}{3} \left[e^{M(c,c)} + e^{M(d,d)} \right] + \frac{1}{6} \left[e^{M(c,d)} + e^{M(d,c)} \right]$$

where $M(c, d) = F(c) + G(d)$.

Proof. As F and G are exponentially convex functions,

$$e^{F((1-s)c+sd)} \leq (1-s)e^{F(c)} + se^{F(d)} \; \forall c, d \in \mathcal{K}; s \in [0, 1]$$

$$e^{G((1-s)c+sd)} \leq (1-s)e^{G(c)} + se^{G(d)} \; \forall c, d \in \mathcal{K}; s \in [0, 1] \; ,$$

multiplying the above two inequalities, we get

$$
\begin{aligned}
0 \; &\leq \; e^{F((1-s)c+sd)}.e^{G((1-s)c+sd)} \\
&\leq \; (1-s)^2 e^{F(c)+G(c)} + s(1-s)e^{F(c)+G(d)} + (s)^2 e^{F(d)+G(d)} + s(1-s)e^{F(d)+G(c)} \; .
\end{aligned}
$$

Put $x = (1-s)c + sd$ and integrating over $[0, 1]$, we obtain:

$$\frac{1}{d-c} \int_c^d e^{F(x)+G(x)} dx \leq \frac{1}{3} \left[e^{F(c)+G(c)} + e^{F(d)+G(d)} \right] + \frac{1}{6} \left[e^{F(c)+G(d)} + e^{F(d)+G(c)} \right]$$

$$\implies \; \frac{1}{d-c} \int_c^d e^{F(x)+G(x)} dx \leq \frac{1}{3} \left[e^{M(c,c)} + e^{M(d,d)} \right] + \frac{1}{6} \left[e^{M(c,d)} + e^{M(d,c)} \right] \; .$$

$\square$

5. PRODUCT OF EXPONENTIAL HARMONIC CONVEX FUNCTION

In this section we have discussed Hermite Hadamard Integral inequality on product of exponential harmonic convex function.

Theorem 5.1. *Let the mapping $F, G : \mathcal{K} \subset \mathbb{R} \setminus \{0\} \to \mathbb{R} \; a, b \in \mathcal{K}, with a < b$ be harmonic convex functions such that $F, G \in L[a, b]$. If the product F*G is harmonic convex function on $[a, d]$ then ,*

$$\frac{ab}{b-a} \int_a^b \frac{e^{F(x)+G(x)}}{x^2} dx \leq \frac{e^M}{N-M} [e - 1]$$

where $M = F(a) + G(b)$ and $N = F(b) + G(a)$.

Proof. As F and G are harmonic convex functions,

$$e^{F\left(\frac{ab}{ta+(1-t)b}\right)} \le e^{tF(b)+(1-t)F(a)} \, \forall a, b \in \mathcal{K}; t \in [0, 1]$$

$$e^{G\left(\frac{ab}{ta+(1-t)b}\right)} \le e^{tG(b)+(1-t)G(a)} \, \forall a, b \in \mathcal{K}; t \in [0, 1] \, .$$

Multiplying the above two inequalities, we get

$$e^{F\left(\frac{ab}{ta+(1-t)b}\right)} * e^{G\left(\frac{ab}{ta+(1-t)b}\right)} = e^{F\left(\frac{ab}{ta+(1-t)b}\right)+G\left(\frac{ab}{ta+(1-t)b}\right)} \le e^{t[G(b)+F(b)]+(1-t)[F(a)+G(a)]} \, .$$

Put $x = (1-s)c + sd$ and integrating over $[0, 1]$, we obtain

$$\frac{ab}{b-a} \int_a^b \frac{e^{F(x)+G(x)}}{x^2} dx \le e^M \int_0^1 e^{t[N-M]} dt$$

$$= \frac{e^M}{N-M}[e-1]$$

$$\implies \frac{ab}{b-a} \int_a^b \frac{e^{F(x)+G(x)}}{x^2} dx \le \frac{e^M}{N-M}[e-1] \, .$$

$\square$

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