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TWO THETA-FUNCTION IDENTITIES OF LEVEL 10

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ABSTRACT. In this paper, we prove two theta-function identities using modular equation of degree 3. Furthermore, as an application of this we establish combinatorial interpretations of colored partitions.

1. INTRODUCTION

Ramanujan's theta function $\mathfrak{f}(x, y)$ is defined as

$$\mathfrak{f}(x, y) := \sum_{n=-\infty}^{\infty} x^{n(n+1)/2} y^{n(n-1)/2} \qquad |xy| < 1.$$

The function $f(x, y)$ enjoys the well-known Jacobi's triple-product identity [5, p. 35] given by

$$\mathfrak{f}(x, y) = (-x; xy)_\infty (-y; xy)_\infty (xy; xy)_\infty \,,$$

where here and throughout the paper, we assume $|q| < 1$ and employ the standard notation

$$(x; q)_\infty := \prod_{n=0}^{\infty} (1 - xq^n).$$

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The important special cases of $\mathfrak{f}(x,y)$ [5, p. 36] are as follows:

$$\psi(q) := \mathfrak{f}(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

$$\varphi(q) := \mathfrak{f}(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty,$$

$$f(-q) := \mathfrak{f}(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

After Ramanujan, we define

$$\chi(q) := (-q;q^2)_\infty.$$

For convenience we write $f(-q^n) = f_n$. Also, one can easily see that

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \qquad \chi(q) = \frac{f_2^2}{f_1 f_4} \qquad \text{and} \qquad \chi(-q) = \frac{f_1}{f_2}. \tag{1.1}$$

A theta function identity which relates f_1, f_2, f_n and f_{2n} is called a theta function identity of level $2n$. Ramanujan documented many theta functions which involve quotients of the function f_1 at different arguments. For example, if [6, p. 206]

$$P := \frac{f_1}{q^{1/6} f_5} \qquad \text{and} \qquad Q := \frac{f_2}{q^{1/3} f_{10}}$$

then

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3.$$

B. C. Berndt [5] proved similar type of identities and used it to evaluate various continued fractions, weber class invariants, theta functions and many more. After the publication of [5,6], many mathematicians discovered similar identities in the spirit of Ramanujan. For the wonderful work, one can see [1–4, 9, 16, 17]. Motivated by the above work, M. Somos [11] used a computer to discover around 6277 new elegant Dedekind eta-function identities of various levels without offering the proof. He runs PARI/GP scripts and it works as a sophisticated programmable calculator. Many authors [12–15, 18] have given the proof of Somos's identities of various levels and found the applications of these in colored partitions. S. Cooper [7, 8] proved some Dedekind eta-function identities of level 6 while finding series and iterations for $1/\pi$. These identities are also recorded by Somos [11] using PARI/GP scripts. Motivated by this, in the present work we prove these identities by using modular equation of degree 3 in Section 2. As an application of this, we obtain interesting combinatorial interpretations of colored partitions in Section 3. Before that we define a modular equation as given in the literature. A modular equation of degree n is an equation relating α and β that is induced by

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}=\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)},$$

where

$${}_2F_1(p,q;r;x):=\sum_{n=0}^{\infty}\frac{(p)_n(q)_n}{(r)_n n!}x^n \qquad |x|<1,$$

denotes an ordinary hypergeometric function with

$$(p)_n:=p(p+1)(p+2)...(p+n-1).$$

Then, we say that β is of degree n over α and call the ratio

$$m:=\frac{z_1}{z_n},$$

the multiplier, where $z_1= {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)$ and $z_n= {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)$.

2. Main results

Theorem 2.1. *We have*

$$\psi^2(q)-q\psi^2(q^5)=\frac{f_2f_5^3}{q^{1/4}f_1f_{10}}.$$

Proof. The following modular equation of degree 5 is recorded by Ramanujan on page 236 of his second notebook [10] and Entry 13(ix) and (xiv) [5, pp. 280 – 288]:

$$1+4^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12}=\frac{m}{2}\left(1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}\right), \tag{2.1}$$

$$1+4^{1/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/12}=\frac{5}{2m}\left(1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}\right) \tag{2.2}$$

and if

$$P:=\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} \qquad \text{and} \qquad Q:=\left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/8}$$

then, we have

$$Q+\frac{1}{Q}+2\left(P-\frac{1}{P}\right)=0, \tag{2.3}$$

where β has degree 5 over α and m is the multiplier of degree 5. From (2.1) and (2.2), we have

$$\frac{m^2}{5} = \frac{1+4^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12}}{1+4^{1/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/12}}. \tag{2.4}$$

From Entry 10(i) and 12(v) [5, pp. 122–124], we have

$$\varphi(q) := \sqrt{z} \tag{2.5}$$

and

$$\chi(q) := 2^{1/6}\left(\frac{x(1-x)}{q}\right)^{-1/24} \tag{2.6}$$

where $q = e^y$ and $y = \pi \ {}_2F_1\left(1/2, 1/2; 1; 1-x\right)/{}_2F_1\left(1/2, 1/2, ; 1; x\right)$. From (2.6), we can write

$$\chi(q) = 2^{1/6}\left(\frac{q}{\alpha(1-\alpha)}\right)^{1/24} \qquad \text{and} \qquad \chi(q^5) = 2^{1/6}\left(\frac{q^5}{\beta(1-\beta)}\right)^{1/24}.$$

From the above, we deduce

$$2^{4/3}q^2\frac{\chi^2(q)}{\chi^{10}(q^5)} = \left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12} \qquad \text{and} \qquad 2^{4/3}\frac{\chi^2(q^5)}{\chi^{10}(q)} = \left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/12}.$$

On using (2.5) and the above, in (2.4), we obtain

$$\frac{\varphi^4(q)}{5\varphi^4(q^5)} = \frac{1+4\frac{q^2\chi^2(q)}{\chi^{10}(q^5)}}{1+4\frac{\chi^2(q^5)}{\chi^{10}(q)}}. \tag{2.7}$$

Similarly, on transcribing (2.3) into theta function, we obtain

$$\frac{u^3}{v^3} + \frac{v^3}{u^3} + \frac{4}{u^3v^3} - u^3v^3 = 0\,, \tag{2.8}$$

where,

$$u := u(q) = q^{-1/24}\chi(q) \qquad \text{and} \qquad v := v(q) = q^{-5/24}\chi(q^5).$$

On multiplying (2.8) throughout by $(uv)^{-10}(u^6 + 4uv + u^5v^5 - v^6)$, we obtain

$$\frac{v^2}{u^{10}} + \frac{16}{v^8u^8} - 1 - \frac{2u}{v^5} - \frac{8}{u^9v^3} - \frac{u^2}{v^{10}} = 0,$$

which is equivalent to

$$\frac{5v^2}{u^{10}}\left(1+\frac{4u^2}{v^10}\right) = \left(1+\frac{u}{v^5}\right)^2\left(1+\frac{4v^2}{u^{10}}\right).$$

Employing (2.7) in the above, we see that

$$\frac{v}{u^5}\frac{\varphi^2(q)}{\varphi^2(q^5)} = 1 + \frac{u}{v^5}. \tag{2.9}$$

From (1.1), we observe

$$\frac{\varphi(q)}{\varphi(q^5)} = \frac{u^2}{q^{1/3}v^2}\frac{f_2}{f_{10}}. \tag{2.10}$$

Using (2.10) in (2.9), we have

$$\frac{q^{-2/3}}{uv^3}\left(\frac{f_2}{f_{10}}\right)^2 - \frac{u}{v^5} - 1 = 0.$$

Letting $q \to -q$ in the above, rewriting $u(-q)$ and $v(-q)$ in terms of f_n by employing (1.1) and then multiplying throughout by $f_1^2 f_5^2 f_{10}$, we deduce the result. $\square$

Theorem 2.2. *We have*

$$\varphi^2(-q^5) - \varphi^2(-q) = \frac{4qf_1f_{10}^3}{f_2f_5}.$$

Proof. On multiplying (2.8), throughout by $4(uv)^{-10}(u^6 + 4uv + u^5v^5 - v^6)$, we obtain

$$4 - \frac{4v^2}{u^{10}} + \frac{8u}{v^5} + \frac{32}{u^9v^3} + \frac{4u^2}{v^{10}} - \frac{64}{v^8u^8} = 0,$$

which can be rewritten as

$$\left(1 - \frac{4u}{v^5}\right)^2\left(1 + \frac{4v^2}{u^{10}}\right) - 5 - \frac{20u^2}{v^{10}} = 0.$$

Employing (2.7) in the above, we see that

$$\left(\frac{v^4}{u^4} + \frac{4}{u^3v}\right)\frac{u^4}{v^4}\frac{\varphi^2(q^5)}{\varphi^2(q)} = 1.$$

Employing (2.10) in the above, we obtain

$$\left(\frac{v^4}{u^4} + \frac{4}{u^3v}\right)\left(\frac{f_{10}}{f_2}\right)^2 = q^{-2/3}.$$

Letting $q \to -q$ in the above, rewriting $u(-q)$ and $v(-q)$ in terms of f_n by employing (1.1) and then multiplying throughout by $f_2^2 f_5 f_{10}^2$, we deduce the result. $\square$

3. Application to Partitions

For simplicity, in sequel we employ the notation

$$(x_1, x_2, \ldots x_n; q)_\infty := (x_1; q)_\infty (x_2; q)_\infty \ldots (x_n; q)_\infty,$$

and define,

$$(q^{r\pm}; q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty, \qquad r, s \in \mathbb{N}; r < s.$$

For example of this, $(q^{3\pm}; q^8)_\infty$ means $(q^3, q^5; q^8)_\infty$, which is $(q^3; q^8)_\infty\,(q^5; q^8)_\infty$. Now we define colored partition as defined as in the literature.

"A positive integer n has l colors if there are l copies of n available colors and all of them are viewed as distinct objects. Partitions of a positive integer into parts with colors are called colored partitions".

For instance if 3 colors are assigned to 1, then the possible colored partitions of 2 are $1_b + 1_b$, $1_y + 1_y$, $1_i + 1_i$, $1_b + 1_y$, $1_b + 1_i$, $1_i + 1_y$ and 2, where we utilized the indices b (blue), y (yellow) and i (indigo) to recognize three colors of 1. Also

$$\frac{1}{(q^a; q^b)_\infty^k},$$

is the generating function for the number of partitions of n where all the parts are congruent to $a \pmod b$ and have k colors. In this section we demonstrate this by giving the combinational interpretations for our results discussed in Section 2.

Theorem 3.1. *Let $\alpha(n)$ represent the number of partitions of n being divided into parts congruent to ± 1 or ± 3 modulo 10 with two colors each and +5 modulo 10 with five colors. Let $\beta(n)$ indicate the number of partitions of n being split into parts congruent to ± 2 or $\pm$ 4 modulo 10 with two colors each. Let $\gamma(n)$ taken to represent the number of partitions of n into several parts congruent to ± 1, ± 3 or +5 modulo 10 with one color each and ± 2 or ± 4 modulo 10 with two colors each, then we have*

$$\alpha(n) - \beta(n-1) - \gamma(n) = 0, \qquad n \geq 1.$$

Proof. On rewriting Theorem 2.1 using (1.1) and then dividing throughout by $f_1^2 f_2^4 f_5^5 f_{10}^5$, we obtain

$$\frac{1}{f_1^2 f_5^3 f_{10}^4} - \frac{q}{f_2^4 f_{10}^5} - \frac{1}{f_1 f_2^3 f_{10}^5} = 0. \tag{3.1}$$

Rewriting the above subject to the common base q^{10}, and using the known fact $(q^{\pm a}; q^b)_\infty = (q^a, q^{b-a}; q^b)_\infty$ for $a, b \in \mathbb{Z}^+$ with $a < b$, (3.1) reduces to

$$\frac{1}{(q_2^{\pm 1}, q_2^{\pm 3}, q_5^{+5}; q^{10})_\infty} - \frac{q}{(q_2^{\pm 2}, q_2^{\pm 4}; q^{10})_\infty} - \frac{1}{(q_1^{\pm 1}, q_2^{\pm 2}, q_1^{\pm 3}, q_2^{\pm 4}, q_1^{+5}; q^{10})_\infty} = 0.$$

The above identity generates $\alpha(n)$, $\beta(n)$ and $\gamma(n)$ as the generating functions and hence, we have

$$\sum_{n=0}^{\infty} \alpha(n) q^n - q \sum_{n=0}^{\infty} \beta(n) q^n - \sum_{n=0}^{\infty} \gamma(n) q^n = 0.$$

Now, on extracting the powers of q^n in the above, we obtain the result. □

Following table verifies the partitions for $n = 2$.

$\alpha(2) = 3$	$1_r + 1_r, 1_y + 1_y, 1_r + 1_y$.
$\beta(1) = 0$	
$\gamma(2) = 3$	$1 + 1, 2_r, 2_m$.

Theorem 3.2. *Let $\alpha(n)$ denote the number of partitions of n into parts congruent to ± 1 or ± 3 modulo 10 with four colors each, ± 2 or ± 4 modulo 10 with two colors each and $+5$ modulo 10 with one color. Let $\beta(n)$ represent the number of partitions of n being divided into parts congruent to $+5$ modulo 10 with one color. Let $\gamma(n)$ indicate the number of partitions of n being split into parts congruent to ± 1 or ± 3 modulo 10 with three colors each, ± 2, or ± 4 modulo 10 with two colors each, then we have*

$$\alpha(n) - \beta(n) - 4\gamma(n-1) = 0, \qquad n \geq 1.$$

Proof. On rewriting Theorem 2.2 using (1.1) and then dividing throughout by $f_1^4 f_2^2 f_5^5 f_{10}^5$, we obtain

$$\frac{1}{f_1^2 f_5^3 f_{10}^4} - \frac{1}{f_2^4 f_{10}^5} - \frac{4q}{f_1 f_2^3 f_{10}^5} = 0. \tag{3.2}$$

Rewriting the above subject to the common base q^{10}, and using the known fact $(q^{\pm a}; q^b)_\infty = (q^a, q^{b-a}; q^b)_\infty$ for $a, b \in \mathbb{Z}^+$ with $a < b$, (3.2) reduces to

$$\frac{1}{(q_4^{\pm 1}, q_2^{\pm 2}, q_4^{\pm 3}, q_2^{\pm 4}, q_1^{+5}; q^{10})_\infty} - \frac{1}{(q_1^{+5}; q^{10})_\infty} - \frac{4q}{(q_3^{\pm 1}, q_2^{\pm 2}, q_3^{\pm 3}, q_2^{\pm 4}; q^{10})_\infty} = 0.$$

The above identity generates $\alpha(n)$, $\beta(n)$ and $\gamma(n)$ as the generating functions and hence, we have

$$\sum_{n=0}^{\infty} \alpha(n) q^n - \sum_{n=0}^{\infty} \beta(n) q^n - q \sum_{n=0}^{\infty} \gamma(n) q^n = 0.$$

Now, on extracting the powers of q^n in the above, we obtain the result. □

Following table verifies the partitions for $n = 2$.

$\alpha(2) = 12$	$1_r+1_r, 1_y+1_y, 1_m+1_m, 1_g+1_g, 1_r+1_y, 1_r+1_m, 1_r+1_g 1_y+1_m$ $1_y + 1_g, 1_m + 1_g, 2_r, 2_g$.
$\beta(1) = 0$	
$\gamma(2) = 3$	$1_y, 1_r, 1_m$.

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