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STABILITY AND BOUNDEDNESS PROPERTIES OF A RATIONAL EXPONENTIAL DIFFERENCE EQUATION

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ABSTRACT. This article aims to discuss, the stability and boundedness character of the solutions of the rational equation of the form

$$y_{t+1} = \frac{\nu\rho^{-y_t} + \delta\rho^{-y_{t-1}}}{\mu + \nu y_t + \delta y_{t-1}}, \quad t \in N(0). \tag{0.1}$$

Here, $\rho > 1, \nu, \delta, \mu \in (0, \infty)$ and y_0, y_1 are taken as arbitrary non-negative reals and $N(a) = \{a, a+1, a+2, \cdots\}$. Relevant examples are provided to validate our results. The exactness is tested using MATLAB.

1. INTRODUCTION

In the past few decades, the theory of difference equations has grown at an accelerated pale. Its importance is very much noticed in applicable analysis and continue to play an active role in mathematics as a whole. Difference equations manifest themselves as mathematical models describing real life situations in probability theory, queuing theory, statistical analysis, stochastic time series, combinatorial analysis, number theory, geometry, electrical networks, genetics in biology, economics, psychology, sociology etc. Usually these are considered as the discrete analogues of differential equations.

Discrete differential equations, normally preserve symmetries. But often, the qualitative properties of solutions of difference equations are quite different

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from those of the corresponding differential equations solutions of several well known difference equations like Clairatu's, Euler's, Riccati's, Bernouilli's Verhulst's Mathieu's and Volterra's difference equations preserve most of the properties of the corresponding differential equations [1–4].

The study on the oscillation and asymptotic properties of solutions of difference equations gain momentum in the last few decades. Heretic research activity happens on this area and lot of research articles are available on the study of the qualitative properties of solutions of difference equations. But it is rare to find articles on the study of properties of solutions like stability and periodicity. Particularly for rational difference equations involving the exponential function, the focus is not much.

Interestingly, many exiting applications are noticed in biology involving difference equations involving exponential terms. To cite an example, the authors [11] studied the oscillatory and chaotic nature of difference equation

$$B_{t+1} = \mu N \frac{e^{\nu-\delta L_t}}{1+e^{\nu-\delta L_t}},\ L_{t+1} = \frac{L_t^2}{L_t+d} + \mu s N \frac{e^{\nu-\delta L_t}}{1+e^{\nu-\delta L_t}}, s \in (0,1). \tag{1.1}$$

Equation (1.1) refers to the evolution of a perennial grass, usually depends on the biomass, the litter mass and the total soil nitrogen. In (1.1), B_t refers to the living mass, L_t, the litter mass, N_t the total soil nitrogen respectively ν, δ, μ, $d > 0$ are fixed positive constants and t refer to time.

El-Metwally et all [7] have investigated the global stability, boundedness and periodicity of the positive difference equation

$$y_{t+1} = \alpha + \beta y_{t-1} e^{-y_t}, \qquad t \in N(0)\,,$$

where α, $\beta > 0$ are the immigration rate and the population growth respectively and the initial conditions y_0 and y_1 are arbitrary nonnegative numbers.

Ozturk et all [8, 9] have studied the boundedness and asymptotic behavior of the difference equations

$$y_{t+1} = \frac{\alpha+\beta e^{-y_t}}{\gamma + y_{t-1}}, and$$

$$y_{t+1} = \frac{\alpha e^{-(t y_t + (t-s) y_{t-s})}}{\beta + y_t + (t-s) y_{t-s}}, \qquad t \in N(0)$$

where α, $\beta > 0$ and $s \in N(1)$ and the initial conditions y_{-j} are reals for $j = 0, 1, 2, \cdots, s$.

G. Papaschinopoulos et all [10] established boundedness and the persistence of the positive solutions, the existence, and the global asymptotic stability of the unique positive equilibrium and the existence of periodic solutions concerning the biological model

$$y_{t+1} = \frac{\nu y_t^2}{y_t + \delta} + \mu \frac{e^{s-dy_t}}{1 + e^{s-dy_t}},$$

where δ, μ, d, s and $0 < a < 1$, are positive constants and y_0 is a real number.

F. Bozkurt in [5] discussed stability behaviour of the equation

$$y_{t+1} = \frac{\alpha e^{-y_t} + \beta e^{-y_{t-1}}}{\gamma + \alpha y_t + \beta y_{t-1}}, t \in N(0). \tag{1.2}$$

Here, the initial conditions are taken as arbitrary reals and α, β are positive numbers.

In this paper, we generalize (1.2) and establish new conditions for stability and other behaviors of the equations (0.1) for $\rho > 1$. MATLAB is used to test the exactness of the behavior of the solutions.

2. Preliminaries

In this section we give some basic definitions and a theorem which is used to prove our main results.

Definition 2.1. *[6] Let $f : I \times I \to I$, $I \in \mathbb{R}$, be a continuous function and $y_0, y_{-1} \in I$ be given values. Then, for*

$$y_{t+1} = g(y_t, y_{t-1}), t \in N(1) \tag{2.1}$$

$\bar{y} \in I$ is called equilibrium of (2.1) if $g(\bar{y}, \bar{y}) = \bar{y}$.

Definition 2.2. *[6] Let $p = \frac{\partial g}{\partial u}(\bar{y}, \bar{y})$ and $q = \frac{\partial g}{\partial v}(\bar{y}, \bar{y})$ denote the partial derivatives of $f(u, v)$ evaluated at an equilibrium $\bar{y}$ of (0.1). Then the equation*

$$y_{t+1} = py_t + qy_{t-1}, t \in N(0) \tag{2.2}$$

is called the linearized equation associated with (0.1) about the equilibrium point $\bar{y}$.

The auxillary equation of (2.2) is the equation

$$\lambda^2 - p\lambda - q = 0 \tag{2.3}$$

with characteristic roots $\lambda_{\pm} = \dfrac{p \pm \sqrt{p^2 + 4q}}{2}$.

Theorem 2.1. *(Linearized stability) [6]*

(i) *If two roots of* (2.3) *are in the region* $|\rho| < 1$*, then we have an equilibrium* $\bar{y}$ *of* (0.1) *which is asymptotically and locally stable.*

(ii) *If at least one of the roots of (2.3) is in the region* $|\rho| < 1$*, then the equilibrium* $\bar{y}$ *of (0.1) is unstable.*

(iii) *The two roots of* (2.3) *will lie in the open region* $|\rho| < 1$ *if and only if*

$$|p| < 1 - q < 2.$$

This locally asymptotically stable equilibrium point $\bar{y}$ *is called a sink.*

(iv) *The magnitude of one of the two roots of (2.3) is more than unity if and only if*

$$|1 - q| > |p| \text{ and } |q| > 1.$$

This equilibrium point $\bar{y}$ *is called a repeller.*

(v) *The absolute value of one of the roots of* (2.3) *is more than unity and the other has absolute value less than unity if and only if*

$$|p| > |1 - q| \text{ and } p^2 + 4q > 0.$$

and this unstable equilibrium point $\bar{y}$ *is called a saddle point.*

(vi) *If a root of (2.3) has absolute value unity, then* $|p| = |1 - q|$ *or* $q = -1$ *and* $|p| \leq 2$*. Conversely, if* $|p| = |1 - q|$ *or* $q = -1$ *and* $|p| \leq 2$ *then we get one root whose absolute value is equal to unity and hence we get the equilibrium point* $\bar{y}$*, which is non hyperbolic.*

3. Main Results

Here, we discuss the existence, uniqueness and stability of the equation (0.1). First, let us prove the existence and uniqueness of solutions of (0.1).
Solving $\bar{y} = \dfrac{(\nu + \delta)\rho^{-\bar{y}}}{\mu + (\nu + \delta)\bar{y}}$ we get equilibrium solutions.
Consider $F(y) = \dfrac{(\nu + \delta)\rho^{-y}}{\mu + (\nu + \delta)y} - y$.

Clearly $F(0) = \dfrac{\nu+\delta}{\mu} > 0$ and $\lim\limits_{y\to\infty} F(y) = -\infty$.
This gives us the existence of equilibrium $\bar{y}$.

$$F'(y) = -\frac{(\nu+\delta)\rho^{-y}\Big(\mu\ln\rho + (\nu+\delta)(y\ln\rho+1)\Big)}{(\mu+(\nu+\delta)y)^2} - 1 < 0\,,$$

which implies that F is decreasing and hence, the equilibrium $\bar{y}$ is unique.

Theorem 3.1. *Equation* (0.1) *has the following properties.*

(i) *Every positive solution of equation* (0.1) *is bounded.*

(ii) *The unique equilibrium point $\bar{y} > 0$ of the equation* (0.1) *is bounded.*

Proof. (i) Let $\{y_t\}$ satisfies equation (0.1) and

$$0 < y_{t+1} = \frac{\nu\rho^{-y_t} + \delta\rho^{-y_{t-1}}}{\mu + \nu y_t + \delta y_{t-1}} < \frac{\nu+\delta}{c}.$$

Hence (i) is true.
(ii) Similarly

$$0 < \bar{y} = \frac{\nu\rho^{-\bar{y}} + \delta\rho^{-\bar{y}}}{\mu + \nu\bar{y} + \delta\bar{y}} < \frac{\nu+\delta}{\mu}.$$

Hence (ii) is true. □

Theorem 3.2. *Let $\delta > \nu$ and*

(3.1)
$$\delta\rho^{-\frac{(2\delta\mu-\nu\mu)\ln\rho}{\nu(\nu-\delta)(\ln\rho+2)-2\delta^2\ln\rho}} < \left(\mu + \frac{((\nu+\delta)\ln\rho-\delta)(2\delta\mu-\nu\mu)\ln\rho}{\nu(\nu-\delta)(\ln\rho+2)-2\delta^2\ln\rho}\right)\ln\rho.$$

Then, the equilibrium point $\bar{y} > 0$ of (0.1) is locally asymptotically stable.

Proof. From Definition 2.2, we get the linearized equation and the characteristic equation associated with (0.1) about $\bar{y}$ is

$$y_{t+1} + \frac{\nu(\rho^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\rho}y_t + \frac{\delta(\rho^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\rho}y_{t-1} = 0, n \in N(0)$$

and

$$\mu^2 + \frac{\nu(\rho^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\rho}\mu + \frac{\delta(\rho^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\rho} = 0, \tag{3.2}$$

respectively. From Theorem 2.1 we obtain

$$\left|-\frac{\nu(\rho^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\rho}\right| < 1 + \frac{\delta(\rho^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\rho} < 2. \tag{3.3}$$

Taking $1+\frac{\delta(\rho^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\rho}<2$ we obtained

$$\rho^{-\bar{y}}<\frac{(\mu+(\nu+\delta)\bar{y})\ln\rho-\delta\bar{y}}{\delta}, \tag{3.4}$$

Taking $\left|-\frac{\nu(\rho^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\rho}\right|<2$ we obtained

$$\rho^{-\bar{y}}<\frac{(\mu+(\nu+\delta)\bar{y})\ln\rho}{\nu-\delta}-\bar{y} \tag{3.5}$$

From (3.3) we arrive

$$\rho^{-\bar{y}}<\frac{2(\mu+(\nu+\delta)\bar{y})\ln\rho-\nu\bar{y}}{\nu}. \tag{3.6}$$

(3.5)⇒ $$(\nu-\delta)\bar{y}<(\mu+(\nu+\delta)\bar{y})\ln\rho+(\delta-\nu)\rho^{-\bar{y}}.$$

Substituting (3.6), we get

$$\bar{y}<\frac{(2\delta\mu-\nu\mu)\ln\rho}{\nu(\nu-\delta)(\ln\rho+2)-2\delta^2\ln\rho}.$$

Again substituting in (3.4), we get

$$\delta\rho^{-\frac{(2\delta\mu-\nu\mu)\ln\rho}{\nu(\nu-\delta)(\ln\rho+2)-2\delta^2\ln\rho}}<\left(\mu+\frac{((\nu+\delta)\ln\rho-\delta)(2\delta\mu-\nu\mu)\ln\rho}{\nu(\nu-\delta)(\ln\rho+2)-2\delta^2\ln\rho}\right)\ln\rho.$$

□

Theorem 3.3. (i) *Equilibrium solution $\bar{y}$ is nonrepeller.*

(ii) *Equilibrium solution $\bar{y}$ is not a saddle point.*

(iii) *Equilibrium solution $\bar{y}$ is a nonhyperbolic point when $\nu\le 2\delta$.*

Proof. (i). From (3.2) and from Theorem 2.1 (iv), we get

$$\left|\frac{\delta(\rho^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\rho}\right|>1 \tag{3.7}$$

and

$$\left|\frac{\nu(\rho^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\rho}\right|<\left|1-\frac{\delta(\rho^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\rho}\right|. \tag{3.8}$$

Substituting (3.7) in (3.8) we get $\nu<0$ which contradicts our assumption that $\nu>0$. Thus the equilibrium solution $\bar{y}$ is nonrepeller.

(ii). From (3.2) and from Theorem 2.1 (v), we get

$$(\mu+(\nu+\delta)\bar{y})\ln\rho>\frac{-\nu^2(\rho^{-\bar{y}}+\bar{y})}{4\delta} \tag{3.9}$$

and

$$\left|\frac{\nu(\rho^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\rho}\right| > \left|1-\frac{\delta(\rho^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\rho}\right|. \tag{3.10}$$

Substituting (3.9) in (3.10) we get $4\nu\delta > \nu^2 + 4\delta^2$.
This is not possible since $\nu > 0$ and $\delta > 0$ are constants. Therefore, no equilibrium solution is a saddle point.
(iii). From (3.2) and from Theorem 2.1 (vi), we get

$$(\mu+(\nu+\delta)\bar{y})\ln\rho = -\delta(\rho^{-\bar{y}}+\bar{y}) \tag{3.11}$$

and

$$\left|\frac{\nu(\rho^{-\bar{y}}+\bar{y})}{(\mu+(\nu+\delta)\bar{y})\ln\rho}\right| \leq 2. \tag{3.12}$$

Substituting (3.11) in (3.12) we get $\nu < 2\delta$. □

4. Examples

In this section we present suitable examples to illustrate our main results.
The following example validates Theorem 3.1.

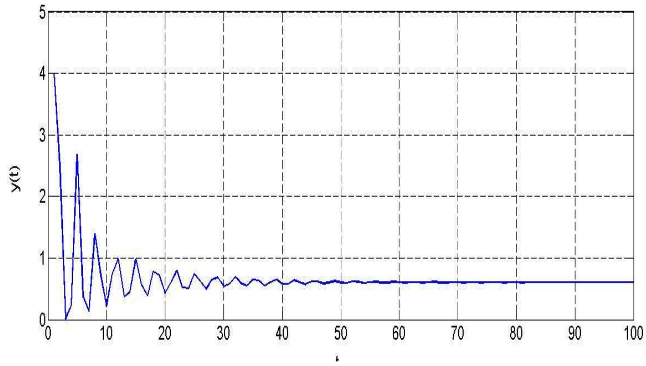
Example 1. *For $\nu = 3, \delta = 5, \mu = 2$ and $\rho = 3, y_{-1} = 4, y_0 = 2.5$, we get $y_{21} = 0.7862 < 4$. In Table 1, numerical values of y_t from the initial values to y_{23} shows they are bounded. Moreover the plot of y_t shown in Figure 1 clears the solutions are all bounded.*

Table 1. Values of y_t from the initial values to y_{23}.

t	0	1	2	3	4	5	6	7
y_t	4.0007	2.5000	0.0086	0.2267	2.6777	0.3632	0.1382	1.4022
t	8	9	10	11	12	13	14	15
y_t	0.7160	0.2164	0.7436	0.9879	0.3712	0.4575	0.9832	0.5587
t	16	17	18	19	20	21	22	23
y_t	0.3866	0.7842	0.7219	0.4291	0.5995	**0.7862**	0.5237	0.5059

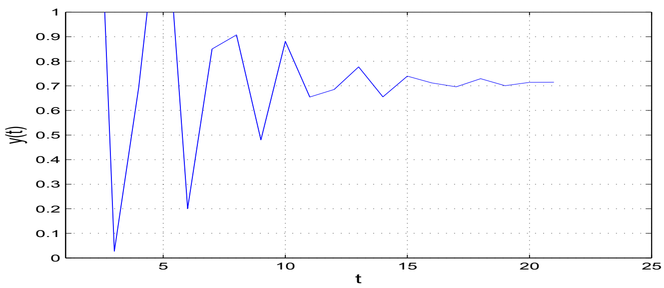
The following example illustrate Theorem 3.2

FIGURE 1. Plot of y_t which shows the boundedness.



Example 2. *For* $\nu = 3, \delta = 5, \mu = 2, \rho = 3$ *and condition (3.1) of the Theorem 3.2 does not hold, then every positive equilibrium solution of (0.1) is not locally asymptotically stable. Moreover, Figure 2 shows the equilibrium solution of* (0.1) *is unstable.*

FIGURE 2. Plot of y_t which shows the unstability.



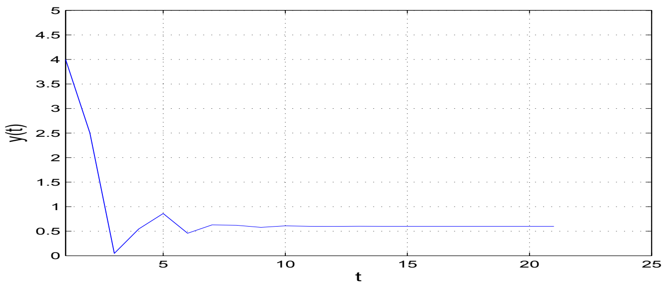
The following example is to illustrate the results of Theorem 3.3

Example 3. *For* $\nu = \delta = \mu$ *and for* $\rho = 2$ *we get equilibrium solution* ≈ 0.65 *which is a nonhyperbolic point. Refer Figule 3.*

5. CONCLUSION

In this paper, we discuss the different characters like stability and boundedness of the solutions of the rational exponential difference equation (0.1) . Earlier results exist for similar type of difference equation when the independent

FIGURE 3. Plot of y_t shows the equilibrium solution is a nonhyperbolic point.



variable is raised as a power of e. Here we have generalized the results when the independent variable is raised to any $\rho > 1$. Earlier results are available only on the study of the stability of the solutions but, we have analyzed more characters like boundedness and the asymptotic behavior of solutions of the equation (0.1) which is new in the literature. Suitable examples are provided to validate our results and they are verified with MATLAB.

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