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OSCILLATION OF A CLASS OF THIRD ORDER GENERALIZED FUNCTIONAL DIFFERENCE EQUATION

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ABSTRACT. This paper aims to study and establish certain criteria on behavior of third order generalized functional difference equations. The authors provide sufficient condition to obtain sequence solution converging to zero to the above said equation. Findings are validated by providing suitable examples.

1. INTRODUCTION

Difference equations and functional equations usually occur due to certain phenomena over time and play essential roles in the field of discrete dynamical systems [1]. Difference equation and their associated operators play a vital role as direct mathematical models of physical phenomena but also provide powerful tools in numerical methods. Difference and its equations also occur in a combined form with differential equations, commonly called differential-difference equations yielding luxurious models, particularly in control theory. Difference equations are widely used in the philosophy of probability, biology, engineering, social and behavioral sciences. Oscillation is one more significant and interest topic of qualitative properties of solutions of certain class of difference-functional-equations. Active research is on in the last few decades in analyzing the solution of equations involving Δ but the study on the same property for

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difference equations involving Δ_ℓ is rare. For the theory related to the relevant topic, one can refer [2], [4], [7], [8], [9], [10].

This research aims to obtain condition for getting oscillatory and convergent solution for the class of 3^{rd} order generalized functional difference equation

$$\Delta_\ell\left(\left[\Delta_\ell\left([\Delta_\ell z(n)]^{\beta_1}\right)a_1(n)\right]^{\beta_2}a_2(n)\right)+q(n)f(x(g(n)))=0, n\geq n_0, \tag{1.1}$$

where $z(n)=x(\tau(n))p(n)+x(n)$. We also present sufficient conditions for sequence solution converging to origin. Here, Δ_ℓ is the forward generalized difference operator defined by $\Delta_\ell y(n)\equiv y(n+\ell)-y(n)=x(n)$, $n\in\mathbb{N}_\ell(n_0)$, $n_0\in[0,\infty)$, $\ell\in(0,\infty)$ and its inverse is defined by

$$y(n)=y(n_0+j)+\sum_{t=0}^{\left[\frac{n-n_0-j-\ell}{\ell}\right]}x(n_0+j+t\ell), \tag{1.2}$$

Consider the notations given below:

(a) $\mathbb{N}_\ell(b)=\{b,\ell+b,2\ell+b,\dots\},\mathbb{N}_1(b)=\mathbb{N}(b)$.
(b) $j=n-n_i-\left[\frac{n-n_i}{\ell}\right]\ell, n_i\in[0,\infty)$, $n_i+j=\bar{n_i}$.
(c) $\{a_i(n)\}$ is a positive increasing sequence and satisfies the condition $\sum\limits_{s=n_0}^{\infty}\frac{1}{a_i^{1/\beta_i}(s)}=\infty$, $i=1,2$ for all $n\geq n_0$.
(d) $0<q(n)$, $p(n)\geq 0, p\in[p(n),1)$.
(e) Integer sequences $\{g(n)\}$ & $\{\tau(n)\}$, $n\geq g(n)$, $\Delta_\ell g(n)>0$, $\lim\limits_{t\to\infty}\tau(t)=\infty$, $\lim\limits_{t\to\infty}g(t)=\infty$.
(f) β_1 and β_2 are odd positive quotients with $\beta=\beta_1\beta_2$.
(g) $0<k\leq\dfrac{f(x)}{x^\beta}$.

2. Basic Definitions and Lemmas

We revisit basic definitions and lemmas to derive our main results.

Lemma 2.1. *[6] Let $\ell\in[0,\infty)$ and $n_\ell^{(\lambda)}=\prod\limits_{t=0}^{\lambda}(n-t\ell)$.Then*

$$\Delta_\ell(n_\ell^{(\lambda)})=(\lambda\ell)n_\ell^{(\lambda-1)}.$$

Lemma 2.2. *[6] If x and y are two real valued functions, then*

$$\Delta_\ell\{x(t)y(t)\}=x(t+\ell)\Delta_\ell y(t)+y(t)\Delta_\ell x(t)\,.$$

Lemma 2.3. *[3] If $u, v > 0$ and $u \neq v$, then*

$$rv^{r-1}(u-v) < u^r - v^r < ru^{r-1}(u-v),\ r < 0, r > 1\,,$$
$$ru^{r-1}(u-v) < u^r - v^r < rv^{r-1}(u-v),\ 0 < r < 1\,.$$

There is obviously equality when $r = 0$, $r = 1$ or $u = v$.

Definition 2.1. *If $x(n)$ satisfies (1.1) and $x(n_2)x(n_2+\ell) \leq 0, n_2 \in N(n_1)$ for any $n_1 \in [a, \infty)$, then it is called oscillatory. Otherwise non oscillatory.*

3. Preliminaries

We establish in this section, oscillation and convergence criteria to (1.1). The following notations are introduced.

$$E_0(n) = z(n),\ E_i(n) = a_i(n)\left(\Delta_\ell E_{i-1}(n)\right)^{\beta_i}, \quad i = 1, 2$$

$$R_N(n) = \frac{1}{a_1^{1/\beta_1}(n)}\left(\sum_{r=0}^{\frac{n-N-\ell-j}{\ell}} \frac{1}{\left[a_2(\bar{N}+r\ell)\right]^{1/\beta_2}}\right)^{1/\beta_1} \quad \text{and}$$

$$\overline{R_N}(n) = \sum_{r=0}^{\frac{n-N-\ell-j}{\ell}} R_N(\bar{N}+r\ell).$$

Lemma 3.1. *If $\{x(t)\}$ is a positive function of solution of (1.1), then for large t,*

(i) $z(t) > 0$, $\Delta_\ell z(t) > 0$ *and* $\Delta_\ell E_1(t) > 0$,
(ii) $z(t) > 0$, $\Delta_\ell z(t) < 0$ *and* $\Delta_\ell E_1(t) > 0$.

Proof. Consider a positive function of solution of (1.1) and $\exists\ n_1 \geq n_0$ such that $0 < x(t)$, $0 < x(\tau(t))$ and $0 < x(g(t))$, $t \geq t_1$. Then $0 < z(t)$ and equation (1.1) yields

$$\Delta_\ell E_2(t) = -f(x(g(t)))q(t) \leq 0.$$

Hence, $E_2(t)$ is a non increasing function and it is positive or negative eventually. We shall show that $0 < E_2(t)$ for $t_1 \leq t$. Suppose that $E_2(t) < 0$, $t_1 \leq t_2 \leq t$, there exists $K_1 > 0$ for $t_2 \leq t_3$, we have

$$\Delta_\ell E_1(t) < -K_1\left[a_2(t)\right]^{-1/\beta_2} < 0,\ \text{for } t \geq t_3.$$

Hence, by equation (1.2)

$$E_1(t) \le E_1(t_3+j) - \sum_{r=0}^{\frac{t-t_3-\ell-j}{\ell}} \frac{K_1}{[a_2(t_3+j+r\ell)]^{1/\beta_2}}.$$

Letting $t \to \infty$, from (c) we have $\lim_{t\to\infty} E_1(t) = -\infty$, $\exists$ a $t_3 \le t_4$ and $K_2 > 0$ and $\Delta_\ell z(t) < -K_2\,[a_1(t)]^{-1/\beta_1}, t \in \mathbb{N}(t_0)$. Adding from t_4 to t, we get

$$z(t) \le z(t_4+j) - \sum_{r=0}^{\frac{t-t_4-\ell-j}{\ell}} \frac{K_2}{[a_1(t_4+j+r\ell)]^{1/\beta_1}}.$$

Allowing $t \to \infty$ and using condition (c), give $z(t) \to -\infty$. That is $z(t) < 0$ eventually which is contradictory to $z(t) > 0$. Therefore $\Delta_\ell E_1(t)$ is positive, that is $\Delta_\ell E_1(t) > 0$ holds. It can be shown from $\Delta_\ell E_1(t) > 0$ that, $\Delta_\ell z(t)$ is monotonically increasing sign in the interval $[t_1, \infty)$, therefore $\Delta_\ell z(t)$ is either negative or positive, which yields (i) and (ii). □

Lemma 3.2. *Let x be a positive function which is a solution of (1.1), and satisfies the condition (ii) of Lemma 3.1. If*

(3.1)

$$\sum_{t=0}^{\infty} \frac{1}{a_1^{1/\beta_1}(\bar{n_3}+t\ell)} \left[\sum_{s=0}^{\frac{t-n_2-\ell-j}{\ell}} \frac{1}{a_2^{1/\beta_2}(\bar{n_2}+s\ell)} \left[\sum_{r=0}^{\frac{s-n_1-\ell-j}{\ell}} q(\bar{n_1}+r\ell) \right]^{\frac{1}{\beta_2}} \right]^{\frac{1}{\beta_1}} = \infty,$$

then $x(n) \to 0$ and $z(n) \to 0$ as $n \to \infty$.

Proof. From the given condition, we have $\lim_{n\to\infty} z(n) = \gamma \ge 0$. We prove that $\gamma = 0$. Suppose that, then $\gamma > 0$, and for any $\epsilon > 0$, $\gamma < z(n) < \gamma + \epsilon$ eventually for sufficiently large n. Choose $0 < \epsilon < \dfrac{1-p}{p}\gamma$. Then we have

$$x(n) = z(n) - x(\tau(n))p(n) > \gamma - z(\tau(n))p(n) > L(\gamma+\epsilon) > Lz(n),$$

where $L = \dfrac{\gamma - p(\gamma+\epsilon)}{\gamma+\epsilon} > 0$. Hence, from equation (1.1) and (g), we have

$$\Delta_\ell E_2(n) \le -(g(n))kq(n)x^\beta < -(g(n))kL^\beta q(n)z^\beta < -q(n)kL^\beta\gamma^\beta\,.$$

Therefore, by equation (1.2), summing this inequality form n_1 to $n-\ell$, we get

$$\Delta_\ell E_1(n) > \frac{(k^{\frac{1}{\beta}} L\gamma)^{\frac{\beta}{\beta_2}}}{a_2^{1/\beta_2}(n)} \left(\sum_{r=0}^{\frac{n-n_1-\ell-j}{\ell}} q(n_1+j+r\ell) \right)^{1/\beta_2}.$$

Summing again form n_2 to $n-\ell$, we obtain

$$\Delta_\ell z(n) < \frac{-C}{a_1^{1/\beta_1}(n)} \left(\sum_{s=0}^{\frac{n-n_2-\ell-j}{\ell}} \frac{1}{a_2^{1/\beta_2}(\bar{n_2}+s\ell)} \left(\sum_{r=0}^{\frac{s-n_1-\ell-j}{\ell}} q(n_1+j+r\ell) \right)^{1/\beta_2} \right)^{1/\beta_1},$$

where $C = k^{\frac{1}{\beta}} L\gamma$. If we add the all the above inequalities, we will get

$$z(n) < -C \sum_{t=0}^{\infty} \frac{1}{a_1^{1/\beta_1}(n_3+j+t\ell)} \left[\sum_{s=0}^{\frac{t-n_2-\ell-j}{\ell}} \frac{1}{a_2^{1/\beta_2}(\bar{n_2}+s\ell)} \left[\sum_{r=0}^{\frac{s-n_1-\ell-j}{\ell}} q(n_1+j+r\ell) \right]^{\frac{1}{\beta_2}} \right]^{\frac{1}{\beta_1}},$$

which contradicts (3.1). This complete the proof. □

Lemma 3.3. *Assume the property (i) of Lemma 3.1 and $z(n) > 0$ be a function of solution of equation (1.1). Then we have*

$$\Delta_\ell E_2(n) \le -kq(n) z^\beta(g(n))(1-p(g(n)))^\beta, \tag{3.2}$$

$$\Delta_\ell z(g(n)) \ge E_2^{1/\beta}(n) R_{n_0}(g(n)) \tag{3.3}$$

and

$$\overline{R}_{n_0}^{\beta}(g(n)) \frac{E_2(n)}{z^\beta(g(n))} \le 1. \tag{3.4}$$

Proof. Consider the given condition on $x(n)$ and the equation (1.1). From (e), $x(n) < 0$, $x(\tau(n)) < 0$ and $x(g(n)) < 0$, $n_0 \le n_1 \le n$. The property (i) in Lemma 3.1 yields $z(n)(1-p(n)) \le x(n) = z(n) - x(\tau(n))p(n)$. Thus, from equation (1.1) and (g), we have

$$\Delta_\ell E_2(n) \le -x^\beta(g(n)) kq(n) \le -z^\beta(g(n)) kq(n)(1-p(g(n)))^\beta < 0.$$

Again, from property (i), there exists an $N \ge n_0$ with

$$E_1(n) = E_1(\bar{N}) + \sum_{r=0}^{m} \frac{E_2^{1/\beta_2}(\bar{N}+r\ell)}{\left[a_2(\bar{N}+r\ell)\right]^{1/\beta_2}}.$$

Where $m = \frac{n-N-\ell-j}{\ell}$, since $\Delta_\ell E_2(n) < 0$, we obtain

$$E_1(n) \geq E_2^{1/\beta_2}(n) \sum_{r=0}^{\frac{n-N-\ell-j}{\ell}} \frac{1}{\left[a_2(\bar{N}+r\ell)\right]^{1/\beta_2}}.$$

This implies that

$$\Delta_\ell z(n) \geq E_2^{1/\beta}(n) R_N(n). \tag{3.5}$$

Since $n \geq g(n)$, leads

$$E_2^{1/\beta}(n) R_N(g(n)) \leq \Delta_\ell z(g(n)).$$

By taking summation in equation (3.5) and using $\Delta_\ell E_2(n) < 0$, yields

$$z(n) \geq z(\bar{N}) + E_2^{1/\beta}(n) \sum_{r=0}^{m} R_N(\bar{N}+r\ell).$$

Where $m = \frac{n-N-\ell-j}{\ell}$, which implies

$$z(n) \geq \overline{R_N}(n) E_2^{1/\beta}(n).$$

Thus, we get

$$z(g(n)) \geq E_2^{1/\beta}(n) \overline{R}_N(g(n)),$$

and so

$$\overline{R}_N^\beta(g(n)) \frac{E_2(n)}{z^\beta(g(n))} \leq 1,$$

which is the required inequality. □

Remark 3.1. *The following notations can be considered for further derivations.*

$$P = \liminf_{n\to\infty} \overline{R}_{n_0}^\beta(g(\bar{n}+\ell)) \sum_{s=n}^{\infty} \phi(\bar{n}+s\ell) \quad \textit{and}$$

and

$$Q = \limsup_{n\to\infty} \frac{\sum_{s=0}^{\left[\frac{n-n_0-j-\ell}{\ell}\right]} \overline{R}_{n_0}^{\beta+1}(g(\bar{n_0}+s\ell+\ell))\phi(\bar{n_0}+s\ell)}{\overline{R}_{n_0}(g(\bar{n}+\ell))},$$

where $\phi(n) = kq(n)(1-p(g(n)))^\beta$. Moreover for $z(n)$ satisfying property (i), we define

$$\omega(n) = \frac{E_2(n)}{z^\beta(g(n))} \tag{3.6}$$

and

$$l = \liminf_{n\to\infty} \overline{R}^{\beta}_{n_0}(g(\bar{n}+\ell))\omega(\bar{n}+\ell). \tag{3.7}$$

$$U = \limsup_{n\to\infty} \overline{R}^{\beta}_{n_0}(g(n+\ell))\omega(n). \tag{3.8}$$

Lemma 3.4. *Let $x(n) > 0$ and be a solution of (1.1).*

(1) *If $P < \infty$, $Q < \infty$, $z(n)$ holds property (i) of Lemma 3.1 and*

$$\lim_{n\to\infty} \overline{R}_{n_0}(n) = \infty, \tag{3.9}$$

then

$$P \leq l - \beta l^{\frac{1+\beta}{\beta}} \text{ and } P + Q \leq 1 \tag{3.10}$$

(2) *$z(n)$ does not hold property (i) if either P (or) $Q = \infty$.*

Proof. **Part(1).** From equation (3.6) and given condition, it is easy to obtain

$$\Delta_\ell \omega(n) = \frac{\Delta_\ell E_2(n)}{z^\beta(g(n))} - \frac{(\Delta_\ell z^\beta(g(n)))E_2(n+\ell)}{z^\beta(g(n+\ell))z^\beta(g(n))}. \tag{3.11}$$

Now, by using equation (3.3), we find that

$$\Delta_\ell z^\beta(g(n)) < \beta z^{\beta-1}(g(n+\ell))\Delta_\ell z(g(n)).$$

The equation (3.11) leads

$$\Delta_\ell \omega(n) = \frac{\Delta_\ell E_2(n)}{z^\beta(g(n))} - \frac{\Delta_\ell z(g(n))\beta E_2(n+\ell)}{z(g(n+\ell))z^\beta(g(n))}.$$

Thus, from (3.2) and (3.3), there exists an $N \geq n_0$ with the condition

$$\Delta_\ell \omega(n) \leq -(1-p(g(n)))^\beta k q(n) - \frac{\beta E_2^{\frac{1+\beta}{\beta}}(n+\ell)R_N(g(n+\ell))}{z^{1+\beta}(g(n+\ell))}, n \geq N$$

This leads to get

$$\Delta_\ell \omega(n) \leq -\phi(n) - \beta R_N(g(n+\ell))\omega^{\frac{1+\beta}{\beta}}(n+\ell). \tag{3.12}$$

From (3.4), we get

$$\overline{R}^{\beta}_N(g(n))\omega(n) \leq 1,$$

which with (3.9) gives

$$\lim_{n\to\infty} \omega(n) = 0. \tag{3.13}$$

From equations (3.6), (3.7) and (3.8), we see that

$$0 \leq l \leq U \leq 1. \tag{3.14}$$

Next, we shall prove the first inequality in (3.10). There exists an $\epsilon > 0$ for sufficiently large integer $n_2 \geq N$ and from the definitions of P and l, we have

$$\overline{R}_N^{\beta}(g(\bar{n}+\ell))\sum_{s=0}^{\infty}\phi(\bar{n}+s\ell) \geq P-\epsilon \quad \text{and}$$

$$\overline{R}_N^{\beta}(g(\bar{n}+\ell))\omega(\bar{n}+\ell) \geq l-\epsilon \quad \text{for } n \geq n_2.$$

By summing (3.12) from n to ∞ and using (3.13), we have

$$\omega(\bar{n}) \geq \sum_{s=0}^{\infty}\phi(\bar{n}+s\ell)+\beta\sum_{s=0}^{\infty}R_N(g(\bar{n}+s\ell+\ell))\omega^{\frac{1+\beta}{\beta}}(\bar{n}+s\ell+\ell). \tag{3.15}$$

Multiplying the above inequality by $\overline{R}_N^{\beta}(g(\bar{n}+\ell))$, we obtain

$$\begin{aligned}
&\omega(\bar{n})\overline{R}_N^{\beta}(g(\bar{n}+\ell)) \geq \omega(\bar{n}) \\
&\geq (P-\epsilon)+(l-\epsilon)^{\frac{1+\beta}{\beta}}\beta\overline{R}_N^{\beta}(g(\bar{n}+\ell))\sum_{s=0}^{\infty}\frac{R_N(g(\bar{n}+s\ell+\ell))}{\overline{R}_N^{1+\beta}(g(\bar{n}+s\ell+\ell))}. \\
&\geq (P-\epsilon)+\beta(l-\epsilon)^{\frac{1+\beta}{\beta}}.
\end{aligned}$$

Taking limit inferior as $n \to \infty$ on both sides, we obtain

$$l \geq (P-\epsilon)+\beta(l-\epsilon)^{\frac{1+\beta}{\beta}}.$$

As $\epsilon \to 0$, we get

$$P \leq l-\beta l^{\frac{1+\beta}{\beta}}.$$

Now, we proceed to prove $P+Q \leq 1$. Multiplying (3.12) by $\overline{R}_N^{\beta+1}(g(n+\ell))$ and adding for n_2 to $n-\ell$, leads

$$\begin{aligned}
&\sum_{s=0}^{m}\overline{R}_N^{\beta+1}(g(\bar{n_2}+s\ell+\ell))\Delta_\ell\omega(\bar{n_2}+s\ell) \\
\leq\ & -\sum_{s=0}^{m}\phi(\bar{n_2}+s\ell)\overline{R}_N^{\beta+1}(g(\bar{n_2}+s\ell+\ell)) \\
&-\beta\sum_{s=0}^{m}\left(\left(\overline{R}_N^{\beta}(g(\bar{n_2}+s\ell+\ell))\omega(\bar{n_2}+s\ell+\ell)\right)^{\frac{1+\beta}{\beta}}\right)(R_N(g(\bar{n_2}+s\ell+\ell)).
\end{aligned}$$

when $m = \frac{n-n_2-j-\ell}{\ell}$. From the product formula for sum of two functions, By Summation by parts, we obtain

$$\begin{aligned}
&\overline{R}_N^{\beta+1}(g(n+\ell))\omega(n) \le \overline{R}_N^{\beta+1}(g(\bar{n_2}+\ell))\omega(\bar{n_2})\\
&+\sum_{s=0}^{m}\omega(\bar{n_2}+s\ell+\ell)\Delta_\ell\overline{R}_N^{\beta+1}(g(\bar{n_2}+s\ell+\ell))\\
&-\sum_{s=0}^{m}\overline{R}_N^{\beta+1}(g(\bar{n_2}+s\ell+\ell))\phi(\bar{n_2}+s\ell)\\
&-\beta\sum_{s=0}^{m}\left(\left(\overline{R}_N^{\beta}(g(\bar{n_2}+s\ell+\ell))\omega(\bar{n_2}+s\ell+\ell)\right)^{\frac{1+\beta}{\beta}}\right)(R_N(g(\bar{n_2}+s\ell+\ell)).\\
&\le \omega(\bar{n_2})\overline{R}_N^{\beta+1}(g(\bar{n_2}+\ell)) - \sum_{s=0}^{\frac{n-n_2-j-\ell}{\ell}}\phi(\bar{n_2}+s\ell)\overline{R}_N^{\beta+1}(g(\bar{n_2}+s\ell+\ell))\\
&+\sum_{s=0}^{\frac{n-n_2-j-\ell}{\ell}}R_N(g(\bar{n_2}+s\ell+\ell))\left((\beta+1)M-\beta M^{\frac{1+\beta}{\beta}}\right),
\end{aligned}$$

where $M = \overline{R}_N^{\beta}(g(\bar{n_2}+s\ell+\ell))\omega(\bar{n_2}+s\ell+\ell)$. Using the inequality

$$Au - Bu^{\frac{1+\beta}{\beta}} \le \frac{A^{1+\beta}}{B^{\beta}}\frac{\beta^{\beta}}{(1+\beta)^{1+\beta}} \tag{3.16}$$

with $u = M$, $A = (1+\beta)$ and $B = \beta$, we obtain

$$\begin{aligned}
&\omega(n)\overline{R}_N^{\beta+1}(g(n+\ell)) \le \omega(\bar{n_2})\overline{R}_N^{\beta+1}(g(\bar{n_2}+\ell))\\
&-\sum_{s=0}^{\frac{n-n_2-j-\ell}{\ell}}\overline{R}_N^{\beta+1}(g(\bar{n_2}+s\ell+\ell))\phi(\bar{n_2}+s\ell)+\overline{R}_N(g(\bar{n}+\ell)).
\end{aligned}$$

It follows

$$\begin{aligned}
&\omega(n)\overline{R}_N^{\beta}(g(n+\ell)) \le \frac{\omega(\bar{n_2})\overline{R}_N^{\beta+1}(g(\bar{n_2}+\ell))}{\overline{R}_N(g(\bar{n}+\ell))}\\
&-\frac{1}{\overline{R}_N(g(\bar{n}+\ell))}\sum_{s=0}^{\frac{n-n_2-j-\ell}{\ell}}\overline{R}_N^{\beta+1}(g(\bar{n_2}+s\ell+\ell))\phi(\bar{n_2}+s\ell)+1.
\end{aligned}$$

Taking limit superior on both sides as $n \to \infty$ and using (3.9) we get

$$U \le 1 - Q.$$

Thus, from (3.14), we see that

$$P \leq l - \beta l^{\frac{1+\beta}{\beta}} \leq l \leq U \leq 1 - Q. \tag{3.17}$$

Thus we have proved (3.10).

Part (2). Suppose that $x(n) > 0$. We shall prove that property (i) will not be satisfied by $z(n)$. Moreover, it's assumed that $P = \infty$. Then, from (3.15), we have

$$\omega(n)\overline{R}_N^{\beta}(g(n+\ell)) \geq \sum_{s=n}^{\infty} \phi(\bar{n}+s\ell)\overline{R}_N^{\beta}(g(n+\ell))$$

Taking limit inferior on both sides as $n \to \infty$, we obtain because of (3.14) that $1 \geq l \geq P = \infty$. This is a contradiction. So we consider the case $Q = \infty$. Then by (3.17), $U = -\infty$, which contradicts with the inequality (3.14). This completes the proof. □

4. Main Results

Theorem 4.1. *Assume that (3.1) and (3.9) hold. If*

$$\liminf_{n\to\infty} \overline{R}_{n_0}^{\beta}(g(\bar{n}+\ell)) \sum_{s=n}^{\infty} \phi(\bar{n}+s\ell) > \frac{1}{(\beta+1)^{\beta+1}}, \tag{4.1}$$

then the solution $\{x(n)\}$ *is either oscillatory or* $\lim_{n\to\infty} x(n) = 0$.

Proof. Suppose that $\{x(n)\}$ is a non oscillatory solution of (1.1) and $x(n)$ is positive. If $P = \infty$, then Lemma 3.4, the property (i) can not be satisfied by $\{x(n)\}$. That is, $z(n)$ satisfies property (ii). Hence, by Lemma 3.2, $x(n) \to 0$ as $n \to \infty$.

Now, Assume $P < \infty$. By Lemma 3.1, we have that $z(n)$ either satisfies the property (i) or the property (ii). If $z(n)$ satisfies property (ii), from Lemma 3.2, we obtain $\lim_{x\to\infty} x(n) = 0$.

Finally, suppose that $z(n)$ holds property (i). From equations (3.6) and (3.7), and Lemma 3.4, we have

$$P \leq l - \beta l^{\frac{\beta+1}{\beta}}.$$

By inequality (3.16) with $u = l$ and $A = 1 = B$,

$$P \leq \frac{1}{(1+\beta)^{1+\beta}},$$

which is a contradiction to inequality (4.1). Hence, the theorem is proved. □

Example 1. *Consider the third order functional ℓ-difference equation*

$$\Delta_\ell\left(\frac{1}{n}\Delta_\ell\left(\frac{1}{n}\Delta_\ell\left(x(n)+\frac{1}{2}x(n-2\ell)\right)\right)\right)$$
$$+\frac{3(4n^3+10n^2\ell+7n\ell^2+2\ell^3)}{n^2(n+\ell)^2(n+2\ell)}x(n-2\ell)=0. \tag{4.2}$$

Since $\beta = 1$ and $f(x) = x$, by Theorem 4.1, (4.2) is oscillatory. Clearly, an oscillatory solution of (4.2) *is* $\{x(n)\} = \left\{(-1)^{\left[\frac{n}{\ell}\right]}\right\}$.

Theorem 4.2. *Suppose that conditions (3.1) and (3.9) hold. If*

$$1 < P + Q,$$

then $\{x_n\}$ is either oscillatory or $\lim_{n\to\infty} x(n) = 0$.

Proof. Suppose that $\{x(n)\} > 0$ is a solution of (1.1). If either P or Q assumes infinity, then $z(n)$ will not satisfy the property (i) of Lemma 3.4, that is, the property (ii) of Lemma 3.1 has to be satisfied by $z(n)$. Then $\lim_{n\to\infty} x(n) = 0$ follows from Lemma 3.2.

Next, suppose that P and Q are finite. Next we show that, either $z(n)$ satisfies property (i) or property (ii) by Lemma 3.1. If $z(n)$ satisfies property (ii), then continuing as above and by Lemma 3.2, we obtain $x(n) \to 0$ as $n \to \infty$.

Finally, suppose that for $z(n)$ satisfies property (i). From Lemma 3.4, we obtain the inequality $P + Q \le 1$, which is a contradiction to the inequality (4.1) and proof is completed. □

Example 2. *The third order functional ℓ-difference equation*

$$\Delta_\ell\left(n\Delta_\ell\left(\Delta_\ell\left(x(n)+\frac{n-\ell}{2n}x(n-\ell)\right)\right)^3\right)$$
$$+\frac{27\ell^7\left(8n^2+27n\ell+27\ell^2\right)(n-\ell)^3}{n^2\left((n+\ell)(n+2\ell)(n+3\ell)\right)^3}x^3(n-\ell)=0, \tag{4.3}$$

satisfies by hypothesis of Theorem 4.2 which implies that equation (4.3) has a solution converging to 0. Clearly, $\{x(n)\} = \left\{\frac{\ell}{n}\right\}$ is one such solution.

Corollary 4.1. *Assume that conditions (3.1) and (3.9) hold, if $Q > 1$. Then $\{x(n)\}$ is either converging to 0 or oscillatory.*

Example 3. *Here, we give an illustration with the equation given below.*

$$\Delta_\ell\left(\Delta_\ell\left(\Delta_\ell\left(x(n)+\frac{1}{3}x(n-\ell)\right)\right)^3\right)+q(n)f(x(g(n)))=0, n>\ell. \tag{4.4}$$

Here $q(n)=\dfrac{32\ell^7\left(16n^4+40n^3\ell+13n^2\ell^2-30n\ell^3+9\ell^4\right)}{3n^2\left((n+\ell)(n+2\ell)(n+3\ell)\right)^3}$, $f(x)=x^3$ *and* $g(n)=n-\ell$. *Then by Corollary 4.1 we obtain* $\lim_{n\to\infty}x(n)=0$. *Infact* $\{x(n)\}=\left\{\dfrac{\ell}{n}\right\}$ *is one such solution of equation (4.4) .*

Theorem 4.3. *Let (3.1) holds. If* $\exists\ \rho(n)>0$ *and*

$$\limsup_{n\to\infty}\sum_{0}^{\frac{n-n_0-j-\ell}{\ell}}\Bigg(\rho(\bar{n_0}+s\ell)\phi(\bar{n_0}+s\ell)$$

$$-\frac{\beta^\beta}{(\beta+1)^{\beta+1}}\left(\frac{\Delta_\ell\rho(\bar{n_0}+s\ell)}{\rho(\bar{n_0}+s\ell)}\right)^{\beta+1}\psi(\bar{n_0}+s\ell)\Bigg)=\infty. \tag{4.5}$$

where $\psi(n)=\rho^{1+\beta}(n+\ell)\rho(n)(\beta R_{n_0}(g(n+\ell)))^{-\beta}$. *Then either* $\{x(n)\}$ *converging to 0 or oscillatory.*

Proof. Suppose $x(n)$ is a bounded and non-oscillatory solution of (4.5) which implies that $x(n)>0$ and $z(n)$ satisfies property (i) or property (ii) by Lemma 3.1. Suppose $z(n)$ satisfies property (ii), then by Lemma 3.2, $x(n)\to 0$ as $n\to\infty$. If $z(n)$ possess property (i), then by Lemma 3.3, we have the inequalities (3.2) and (3.3) hold. Now, we define $\omega_1(n)>0$ as

$$\omega_1(n)=\frac{E_2(n)}{z^\beta(g(n))}\rho(n).$$

By applying Δ_ℓ and using (3.2) and (3.3), we will get the inequality

$$\Delta_\ell\omega_1(n)\le-\rho(n)\phi(n)+\frac{\Delta_\ell\rho(n)}{\rho(n+\ell)}\omega_1(n+\ell)-\psi^{-\frac{1}{\beta}}(n)w_1^{\frac{\beta+1}{\beta}}(n+\ell).$$

Using inequality (3.16) with $u=\omega_1(n+\ell)$, $A=\dfrac{\Delta_\ell\rho(n)}{\rho(n+\ell)}$ and $B=\psi^{-\frac{1}{\beta}}(n)$, we obtain

$$\frac{\Delta_\ell\rho(n)}{\rho(n+\ell)}\omega_1(n+\ell)-\psi(n)^{-\frac{1}{\beta}}w_1^{\frac{\beta+1}{\beta}}(n+\ell)\le\frac{\beta^\beta}{(\beta+1)^{\beta+1}}\left(\frac{\Delta_\ell\rho(n)}{\rho(n+\ell)}\right)^{\beta+1}\psi(n).$$

Therefore, we get

$$\Delta_\ell \omega_1(n) \le -\rho(n)\phi(n) + \left(\frac{\Delta_\ell \rho(n)}{\rho(n+\ell)}\right)^{\beta+1} \psi(n) \frac{\beta^\beta}{(\beta+1)^{\beta+1}}.$$

By adding the above from n_0 to $n-\ell$,

$$\omega_1(n) \le \omega_1(\bar{n_0}) - \sum_{s=0}^{\frac{n-n_0-j-\ell}{\ell}} \Big(\rho(\bar{n_0}+s\ell)\phi(\bar{n_0}+s\ell) - \frac{\beta^\beta}{(\beta+1)^{\beta+1}} \left(\frac{\Delta_\ell \rho(\bar{n_0}+s\ell)}{\rho(\bar{n_0}+s\ell+\ell)}\right)^{\beta+1} \psi(\bar{n_0}+s\ell)\Big).$$

Applying limit superior and using (4.5), leads $\omega_1(n) \to -\infty$, a contradiction to the fact $\omega_1(n) > 0$, which gives the proof. □

Example 4. *For illustration, consider the equation given below.*

$$\Delta_\ell \left(n\Delta_\ell \left(\frac{1}{n} \Delta_\ell \left(x(n) + \frac{1}{2} x(n-3\ell) \right)^5 \right) \right) + \frac{4n^2+10n\ell+5\ell^2}{(n+\ell)(n+2\ell)} x^5(n-4\ell) = 0. \tag{4.6}$$

From Theorem 4.3, we get oscillatory solution. Indeed $\{x(n)\} = \left\{(-1)^{\left[\frac{n}{\ell}\right]}\right\}$ *is one of the oscillatory solution of equation (4.6) .*

5. Conclusion

Using a Riccati type transformation, we have established criteria for the more general 3^{rd} order generalized functional ℓ-difference equation (1.1). We have also set conditions for convergent solution converging to 0. Our results are also the generalization of all the earlier results, especially those of [5]. The technique adopted is also a different form that already existed. The significance of the results is also well established by the examples presented in this paper.

REFERENCES

[1] R. P. AGARWAL: *Difference Equations and Inequalities, Theory, Methods and Applications*, 2nd Edition, Marcel Dekker, New York, 2000.

[2] D. DOSLA, A. KOBZA: *Global Asymptotic Properties of Third-order Difference Equations*, An International Journal Computers and Mathematics with Applications, **48** (2004), 191–200.

[3] G. H. HARDY, J. E. LITTLEWOOD, G. POLYA: *Inequalities*, 2nd Edition, Cambridge University Press, 1952.

[4] J. R. GRAEF, E. THANDAPANI: *Oscillatory and Asymptotic Behavior of Solutions of Third Order Delay Difference Equations*, Funkcialaj Ekvacioj, **42** (1999), 355–369.

[5] K. S. VIDHYAA, C. DHARUMAN, E. THANDAPANI, S. PINELAS: *Oscillatory Theorems for Third Order Nonlinear Delay Difference Equations*, Mathematica Bohemica, **1** (2019), 25–3
.

[6] M. MARIA SUSAI MANUEL, G. BRITTO ANTONY XAVIER, E. THANDAPANI: *Theory of Generalized Difference Operator and its Applications*, Far East Journal of Mathematical science, **20**(2) (2006), 163–171.

[7] B. G. PACHPATTE: *In Equalities for Finite Difference Equations*, Marcl Dekker Inc, New York, Basel, 2002.

[8] J. POPENDA, E. SCHMEIDEL: *Nonoscillatory Solution of Third Order Difference Equations*, Portugaliae Mathematica, **49** (1992), 233–239.

[9] S. R. GRACE, R. P. AGARWAL, J. R. GRAEF: *Oscillation Criteria for Certain Third Order Nonlinear Difference Equations*, Applicable Analysis and Discrete Mathematics, **3**(1) (2009), 27–38.

[10] S. H. SAKER, J. O. ALZABUT, A. MUKHEIMER: *On the Oscillatory Behavior for A Certain Class of Third Order Nonlinear Delay Difference Equations*, Electronic Journal of Qualitative Theory of Differential Equations, **67** (2010), 1-16.

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