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A NOTE ON $*-$REVERSE DERIVATIONS IN RINGS

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ABSTRACT. Let R be a ring with involution $*$ and $\delta : R \to R$ be a $*-$reverse derivation of R. In this note, our intent is to show that a well-known theorem of Bell and Kappe [2, Theorem 3] on derivations, also holds true for $*-$reverse derivations. More specifically, we characterize $*-$reverse derivations of prime and semiprime rings that act as homomorphism or anti-homomorphism on suitable subsets.

1. INTRODUCTION

Let R be an associative ring throughout. A ring R is *prime* (resp. *semiprime*) if for any $a, b \in R$, $aRb = (0)$ (resp. $aRa = (0)$) implies $a = 0$ or $b = 0$ (resp. $a = 0$). An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, an additive mapping $d : R \to R$ satisfying $d(x^2) = d(x)x + xd(x)$ for all $x \in R$ is said to be a Jordan derivation. An additive mapping $F : R \to R$ is called generalized derivation if there exists an associated derivation d such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Clearly, every derivation is a generalized derivation. Herstein [3] introduced the notion of reverse derivation while studying Jordan derivations in prime rings, viz., an additive mapping $\delta : R \to R$ is said to be reverse derivation if $\delta(xy) = \delta(y)x + y\delta(x)$ for all $x, y \in R$. Notice that, every reverse derivation is a Jordan derivation, but the converse is not true in general. In 2015, Aboubakr and González [1]

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extended the notion of reverse derivation to generalized reverse derivation as follows: an additive mapping $F : R \to R$ is called generalized reverse derivation if $F(xy) = F(y)x + y\delta(x)$ for all $x, y \in R$, where δ is a reverse derivation of R. By an involution of R, we mean an additive mapping $* : R \to R$ satisfying: (i) $(xy)^* = y^*x^*$, (ii) $(x^*)^* = x$ for all $x, y \in R$. As usual, $[x, y]$ denotes the commutator $xy - yx$ and the basic commutator identities are $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$.

Very recently, Sandhu et al. [5] introduced the notion of $*-$reverse derivation as follows:

Definition 1.1. *Let R be a ring with involution $*$. By a $*-$*reverse derivation *of a R, we mean an additive mapping $\delta : R \to R$ such that $\delta(xy) = \delta(y)x^* + y^*\delta(x)$ for all $x, y \in R$. If $a \in R$ be a fixed element, then a mapping $x \mapsto [x^*, a]$ is a natural example of $*-$reverse derivation, we call it the* inner $*-$reverse derivation*.*

Example 1. *Let $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ is the ring of integers. Define mappings $\delta, * : R \to R$ such that $\delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -b & 2b \\ a - 2c - d & b \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Clearly, $*$ is an involution on R and δ is a $*-$reverse derivation, which is neither a derivation nor a reverse derivation.*

2. $*-$REVERSE DERIVATIONS AS (ANTI)HOMOMORPHISMS

In a classical paper, Bell and Kappe [2] initiated the study of derivations of prime rings that act as homomorphism or anti-homomorphism on a specific subset of the ring. Precisely, they proved the following result: *Let R be a prime ring and U a nonzero right ideal of R. If d is a derivation of R which acts as a homomorphism or an anti-homomorphism on U, then $d = 0$ on R.* We now show that this result is also valid in $*-$reverse derivation case.

Theorem 2.1. *Let R be a prime ring with involution and λ a nonzero right ideal of R. If R admits a $*-$reverse derivation δ that acts as an anti-homomorphism or a homomorphism on λ, then $\delta = 0$.*

Proof.

(a) Let us assume that $\delta(xy) = \delta(y)\delta(x)$ for all $x, y \in \lambda$. In particular, we have $\delta(x^2y) = \delta(x(xy)) = \delta(y)\delta(x^2)$ for all $x, y \in \lambda$. It implies that

$$\delta(xy)x^* + y^*x^*\delta(x) = \delta(y)\delta(x)x^* + \delta(y)x^*\delta(x),\ \forall x, y \in \lambda.$$

In view of our hypothesis, it follows that

$$(y^* - \delta(y))x^*\delta(x) = 0,\ \forall x, y \in \lambda. \tag{2.1}$$

Replace y by yr in (2.1), where $r \in R$, we obtain

$$r^*(y^* - \delta(y))x^*\delta(x) - \delta(r)y^*x^*\delta(x) = 0.$$

Using (2.1), we get $\delta(r)y^*x^*\delta(x) = 0$ for all $x, y \in \lambda$ and $r \in R$. It implies that $\delta(r)Ry^*x^*\delta(x) = (0)$ for all $x, y \in \lambda$ and $r \in R$. Thus, we conclude that either $\delta = 0$ or $y^*x^*\delta(x) = 0$ for all $x, y \in \lambda$. Let us consider the latter case that $y^*x^*\delta(x) = 0$ for all $x, y \in \lambda$. Now it follows from (2.1) that

$$\delta(y)x^*\delta(x) = 0,\ \forall x, y \in \lambda. \tag{2.2}$$

Let us take xy in place of y in the initial relation, we get $\delta(x(xy)) = \delta(xy)\delta(x)$ for all $x, y \in \lambda$. It implies that

$$\begin{aligned} \delta(x^2y) &= \delta(xy)\delta(x) \\ \delta(y)(x^2)^* + y^*\delta(x^2) &= \delta(y)x^*\delta(x) + y^*(\delta(x))^2. \end{aligned}$$

Using (2.2) and the fact that $\delta(x^2) = (\delta(x))^2$ for all $x \in \lambda$, we left with $\delta(y)(x^*)^2 = 0$ for all $x, y \in \lambda$. Substitute yr instead of y in the last relation, where $r \in R$, we find that $\delta(r)y^*(x^*)^2 = 0$. It implies that $\delta(r)R(x^2y)^* = (0)$ for all $x, y \in \lambda$ and $r \in R$. In view of our assumption, we find that $y^*(x^2)^* = 0$ for all $x, y \in \lambda$. That means, we have $(x^2y)^* = 0$ for all $x, y \in \lambda$. Applying involution, we get $x^2y = 0$ for all $x, y \in \lambda$. In particular, it gives that $x^3 = 0$ for all $x \in \lambda$. In view of Levitzki's result [4, Lemma 1.1], a contradiction follows. It completes our proof.

(b) Suppose that $\delta(xy) = \delta(x)\delta(y)$ for all $x, y \in \lambda$. Replace y by xy, we get

$$y^*x^*\delta(x) = \delta(x)y^*\delta(x),\ \forall x, y \in \lambda. \tag{2.3}$$

Replace y by yr in (2.3), where $r \in R$, we find that

$$r^*y^*x^*\delta(x) = \delta(x)r^*y^*\delta(x). \tag{2.4}$$

Left multiplying (2.3) by r^* and compare with (2.4) in order to get $y^*\delta(x) = 0$ for all $x, y \in \lambda$ and $r \in R$. It implies that $[\delta(x), R]Ry^*\delta(x) = (0)$ for all $x, y \in \lambda$. Since R is a prime ring, for each $x \in \lambda$, either $[\delta(x), R] = (0)$ or $\lambda^*\delta(x) = (0)$. By Brauer's trick, we find that either $[\delta(x), R] = (0)$ for all $x \in \lambda$ or $\lambda^*\delta(x) = (0)$ for all $x \in \lambda$. Let us consider that latter case $y^*\delta(x) = 0$ for all $x, y \in \lambda$. Thus, we see that $\delta(xy) = \delta(y)x^*$ for all $x, y \in \lambda$. By our initial hypothesis, we conclude that $\delta(x)\delta(y) = \delta(y)x^*$ for all $x, y \in \lambda$. Replace x by xr, where $r \in R$, we get $\delta(y)r^*x^* = r^*\delta(x)\delta(y)$. It further implies that $[\delta(\lambda), R]R\lambda^* = (0)$. Since $(0) \neq \lambda^*$, we get $\delta(\lambda) \subseteq Z(R)$. Hence, in each case we have $\delta(x) \in Z(R)$ for all $x \in \lambda$. Therefore, $\delta(xy) = \delta(y)\delta(x)$ for all $x, y \in \lambda$ and the conclusion follows from part (a).

□

Theorem 2.2. *Let R be a semiprime ring with involution and λ a nonzero ideal of R. If R admits a $*-$reverse derivation δ that acts as an anti-homomorphism or a homomorphism on λ, then $\delta|_\lambda = 0$.*

Proof.

(a) Let us assume that $\delta(xy) = \delta(y)\delta(x)$ for all $x, y \in \lambda$. Equivalently, we have

$$\begin{aligned} \delta(y)x^* + y^*\delta(x) &= \delta(y)\delta(x) \\ \delta(y)(x^* - \delta(x)) &= -y^*\delta(x). \end{aligned} \tag{2.5}$$

Replace y by yz in (2.5), we find that $\delta(z)y^*(x^* - \delta(x)) = 0$ for all $x, y, z \in \lambda$. Replace x by rx, we get $\delta(z)y^*x^*\delta(r) = 0$ for all $x, y, z \in \lambda$ and $r \in R$. In particular, we see that $y^*x^*\delta(z)Ry^*x^*\delta(z) = (0)$ for all $x, y, z \in \lambda$. It implies that $y^*x^*\delta(z) = 0$ for all $x, y, z \in \lambda$. In this view, it follows that $x^*\delta(z) = 0$ for all $y, z \in \lambda$. Replace x by rx, where $r \in R$, we find that $x^*R\delta(z) = (0)$ for all $x, z \in \lambda$. Thus, we conclude that $x^*\delta(z) = 0 = \delta(z)x^*$ for all $x, z \in \lambda$. Finally, we see that $\delta(xy) = 0$ for all $x, y \in \lambda$. Substituting ys in place of y in the last relation, we get $\delta(s)(xy)^* = 0$. It implies that $\delta(s)y^*\delta(s)x^* = 0$ for all $x, y \in \lambda$ and $s \in R$. Thus, we conclude that $(\delta(R)\lambda^*)^2 = (0)$ and hence $\delta(R)\lambda^* = (0)$. Likewise, we find $\lambda^*\delta(R) = (0)$. Now, we observe that for any $x, y \in \lambda$ and $r \in R$, we have

$$\begin{aligned} \delta((rx)y) &= \delta(y)(rx)^* + y^*\delta(rx) \\ &= 0. \end{aligned}$$

It implies that $\delta(y)\delta(rx) = 0$ for all $x, y \in \lambda$ and $r \in R$. Replace y by sy, we see that $\delta(y)s^*\delta(rx) = 0$ for all $x, y \in \lambda$ and $r, s \in R$. Take rx in place of y, we conclude that $(R\delta(rx))^2 = (0)$ for all $x \in \lambda$ and $r \in R$. But R has no nonzero nilpotent ideal, hence obtain $\delta(rx) = 0$ for all $x \in \lambda$ and $r \in R$. In the same way, we find that $\delta(xr) = 0$ for all $x \in \lambda$ and $r \in R$. Hence δ vanishes on all sums of the terms rx and xr, with $x \in \lambda$ and $r \in R$. It forces that $\delta = 0$ on λ.

(b) Consider $\delta(xy) = \delta(x)\delta(y)$ for all $x, y \in \lambda$. That means

$$\delta(y)x^* + y^*\delta(x) = \delta(x)\delta(y), \ \forall x, y \in \lambda. \tag{2.6}$$

Replace x by zx in (2.6) in order to get

$$\begin{aligned} \delta(y)x^*z^* + y^*\delta(x)z^* + y^*x^*\delta(z) &= \delta(x)z^*\delta(y) + x^*\delta(z)\delta(y) \\ \delta(x)\delta(y)z^* + y^*x^*\delta(z) &= \delta(x)z^*\delta(y) + x^*\delta(zy) \\ \delta(x)[\delta(y), z^*] &= x^*\delta(zy) - y^*x^*\delta(z). \qquad (2.7) \end{aligned}$$

Replace x by xy in (2.7), we get

$$\delta(y)x^*[\delta(y), z^*] = 0, \ \forall x, y, z \in \lambda.$$

Then it is easy to see that $[\delta(y), z^*]\lambda^*[\delta(y), z^*] = (0)$ for all $x, y, z \in \lambda$. It implies $[\delta(y), z^*] = 0$ for all $y, z \in \lambda$. Replace z by zr, where $r \in R$, we obtain $[\delta(y), r^*]z^* = 0$. Therefore, we have

$$[\delta(\lambda), R]R\lambda^* = (0).$$

Let $\mathfrak{P} = \{P_\alpha : \alpha \in \Lambda\}$ be a family of prime ideals in R such that $\cap P_\alpha = (0)$. From our last expression, we find that for each α, either $\lambda^* \subseteq P_\alpha$ or $[\delta(\lambda), R] \subseteq P_\alpha$. Since $\cap P_\alpha = (0)$ and $\lambda^* \neq (0)$, we have $[\delta(\lambda), R] \subseteq P_\alpha$ for each α. Hence $\delta(\lambda) \subseteq Z(R)$. Further, the conclusion follows from part (a). We are done. $\square$

In this view one can easily deduce the following generalization of Theorem 2 of [2].

Remark 2.1. *Let R be a semiprime ring and λ a nonzero ideal of R. If R admits a derivation δ that acts as a homomorphism or an anti-homomorphism on λ, then $\delta|_\lambda = 0$.*

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