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ULAM STABILITY OF NTH ORDER DIFFERENCE EQUATION

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ABSTRACT. In this article, we enumerate the Hyers-Ulam stability of the nth order difference equation $y_{t+n} = g\left(t, y_t, \ldots, y_{t+n-1}\right)$ by using contraction mapping principle.

1. INTRODUCTION

The Ulam stability problem for various functional equations is invented by a famous talk of S.M. Ulam [19] in 1940. The first positive answer was given by D.H. Hyers [7] in 1941. Since then, many number of researchers have analyzed the Ulam stability problem for various functional equations in different spaces (see [2, 4, 6, 14]).

After that a generalization Ulam's problem was proposed by changing functional equations with differential and difference equations. The Hyers-Ulam stability of differential equations have been established in many papers (see [1, 8, 11, 17]) and the references cited therein. Now a days, only rare results are described in the literature regarding the Hyers-Ulam stability of difference equations (see [3, 5, 10, 12, 13, 15, 16, 18]).

In 2006, S.M. Jung et al. [9] investigate the Hyers-Ulam stability of the first order difference equation. Motivated and connected by the above result, here

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our foremost aim is to establish the Hyers-Ulam stability of the nth order difference equation

$$y_{t+n} = g\left(t, y_t, \ldots, y_{t+n-1}\right) \tag{1.1}$$

for all $t \in \mathbb{N}_0$, by using Banach's contraction mapping theorem.

2. Preliminaries

The following definition is very useful to prove our main results.

Definition 2.1. *Let the nth order difference equation* (1.1) *has the Hyers-Ulam stability if there exists a positive constant $L < 1$ with the following properties: for every $\epsilon > 0$ and for any $\rho : [0, \infty) \to [0, \infty)$ be a monotone increasing function, there exists a $\psi : [0, \infty) \to [0, \infty)$ such that*

$$|p_{t+n} - g(t, p_t, \ldots, p_{t+n-1})| \leq \epsilon$$

for any complex-valued sequence $\{p_t\}_{t\in\mathbb{N}_0}$. Then there exists a complex-valued sequence $\{q_t\}_{t\in\mathbb{N}_0}$ satisfies the difference equation

$$q_{t+n} = g\left(t, q_t, \ldots, q_{t+n-1}\right)$$

and $|q_{t+n-1} - p_{t+n-1}| \leq \psi^{t+n-1}\left(|q_0 - p_0|\right)$ such that $|q_{t+n} - p_{t+n}| \leq L(\epsilon)$. We call such L as the Hyers-Ulam stability constant for the difference equation (1.1).

3. Hyers-Ulam Stability

In this section, we establish the Hyers-Ulam stability of the nth order difference equation (1.1). First, we prove the theorem which will be very useful and powerful tool for proving our main results.

Theorem 3.1. *Given $\epsilon > 0$, let $g : \mathbb{N}_0 \times \mathbb{C}^n \to \mathbb{C}^n$ be a function such that*

$$|g(t, v) - g(t, w)| \leq \rho\left(|v - w|\right) \tag{3.1}$$

where $v = v\left(p_t, \ldots, p_{t+n-1}\right)$ and $w = w\left(q_t, \ldots, q_{t+n-1}\right)$ for all $t \in \mathbb{N}_0$ and all $v, w \in \mathbb{C}^n$, where $\rho : [0, \infty) \to [0, \infty)$ is a monotone increasing function. If a sequence $\{p_t\}_{t\in\mathbb{N}_0}$ which satisfies the inequality

$$|p_{t+n} - g\left(t, p_t, \ldots, p_{t+n-1}\right)| \leq \epsilon \tag{3.2}$$

for all $t \in \mathbb{N}_0$. Then there exists a sequence $\{q_t\}_{t\in\mathbb{N}_0}$ satisfies the difference equation

$$q_{t+n} = g\left(t, q_t, \ldots, q_{t+n-1}\right) \tag{3.3}$$

such that

$$\left|q_{t+n-1} - p_{t+n-1}\right| \leq \psi^{t+n-1}\left(\left|q_0 - p_0\right|\right) \tag{3.4}$$

for all $t \in \mathbb{N}_0$, where the function $\psi : [0,\infty) \to [0,\infty)$ is denoted by $\psi(y) = \rho(y)+\epsilon$, for all $y \geq 0$ and ψ^{t+n-1} denotes the value of $t+n-1^{th}$ iteration of ψ at y.

Proof. By using the principle of recursive definition, there exists a complex-valued sequence $\{q_t\}_{t\in\mathbb{N}_0}$ is uniquely determined by (3.3) provided that q_0 is given.

Now to prove the inequality (3.4), we have to apply an induction method on t. Using (3.1), (3.2) and (3.3) for $t = 0$, we have

$$\begin{aligned}
\left|q_n - p_n\right| &\leq \left|g\left(0, q_0, \ldots, q_{n-1}\right) - p_n\right| \\
&\leq \left|g\left(0, q_0, \ldots, q_{n-1}\right) - g\left(0, p_0, \ldots, p_{n-1}\right)\right| \\
&\quad + \left|g\left(0, p_0, \ldots, p_{n-1}\right) - p_n\right| \\
&\leq \rho\left(\left|q_{n-1} - p_{n-1}\right|\right) + \epsilon \\
\left|q_n - p_n\right| &= \psi^n\left(\left|q_0 - p_0\right|\right).
\end{aligned}$$

Hence by using induction hypothesis, we can assume that the inequality (3.4) is true by putting $t = 1$ in (3.4). Now, we have to prove the inequality (3.4) is true for some $t + n \in \mathbb{N}$. Then, by using (3.1), (3.2) and (3.3), we obtain

$$\begin{aligned}
\left|q_{t+n} - p_{t+n}\right| &\leq \left|g\left(t, q_t, \ldots, q_{t+n-1}\right) - p_{t+n}\right| \\
&\leq \left|g\left(t, q_t, \ldots, q_{t+n-1}\right) - g\left(t, p_t, \ldots, p_{t+n-1}\right)\right| \\
&\quad + \left|g\left(t, p_t, \ldots, p_{t+n-1}\right) - p_{t+n}\right| \\
&\leq \rho\left(\left|q_{t+n-1} - p_{t+n-1}\right|\right) + \epsilon \\
&\leq \psi\left(\psi^{t+n-1}\left(\left|q_0 - p_0\right|\right)\right) \\
&= \psi^{t+n}\left(\left|q_0 - p_0\right|\right)
\end{aligned}$$

This proves the inequality (3.4) for all $t \in \mathbb{N}_0$. □

Now, we have to prove the Hyers-Ulam stability of the nth order difference equation (1.1) under the condition $p_0 = q_0$.

Theorem 3.2. *Given $\epsilon > 0$, let $g : \mathbb{N}_0 \times \mathbb{C}^n \to \mathbb{C}^n$ be a function fulfilling the condition* (3.1) *for all $t \in \mathbb{N}_0$ and $v, w \in \mathbb{C}^n$, and let $\psi : [0,\infty) \to [0,\infty)$be denoted by $\psi(y) = \rho(y) + \epsilon$ for all $y \geq 0$, where $\rho : [0,\infty) \to [0,\infty)$ is a monotone increasing function such that $\rho(0) = 0$ and there exists a positive constant $L < 1$ such that*

$$|\rho(y) - \rho(z)| \leq L\,|y - z|\,, \tag{3.5}$$

for all $y, z \geq 0$. If a complex-valued sequence $\{p_t\}_{t\in\mathbb{N}_0}$ satisfies the inequality (3.2) *for all $t \in \mathbb{N}_0$, then there exists a complex-valued sequence $\{q_t\}_{t\in\mathbb{N}_0}$ satisfies the difference equation* (3.3) *such that*

$$|q_{t+n-1} - p_{t+n-1}| \leq \frac{\epsilon}{1-L} + \frac{L^{t+n-1}}{1-L}\,|\psi\left(|q_0 - p_0|\right) - |q_0 - p_0|| \tag{3.6}$$

for all $t \in \mathbb{N}_0$.

Proof. By applying the principle of recursive definition and using Theorem 3.2, there exists $\{q_t\}_{t\in\mathbb{N}_0}$ satisfying $q_{t+n} = g\left(t, q_t, \ldots, q_{t+n-1}\right)$ and

$$|q_{t+n-1} - p_{t+n-1}| \leq \psi^{t+n-1}\left(|q_0 - p_0|\right), \tag{3.7}$$

for all $t \in \mathbb{N}_0$. It ensue from (3.5) that ψ is also a contraction mapping with a Lipschitz constant L. Then by using the contraction mapping principle, we have

$$\left|\psi^{t+n-1}\left(|q_0 - p_0|\right) - y^*\right| \leq \frac{L^{t+n-1}}{1-L}\,|\psi\left(|q_0 - p_0|\right) - |q_0 - p_0|| \tag{3.8}$$

for all $t \in \mathbb{N}_0$, where y^* is the unique fixed point of ψ, from that we can get

$$\psi^{t+n-1}\left(|q_0 - p_0|\right) \leq y^* + \frac{L^{t+n-1}}{1-L}\,|\psi\left(|q_0 - p_0|\right) - |q_0 - p_0|| \tag{3.9}$$

for all $t \in \mathbb{N}_0$. Using the inequality (3.7) in (3.9), we obtain that

$$|q_{t+n-1} - p_{t+n-1}| \leq y^* + \frac{L^{t+n-1}}{1-L}\,|\psi\left(|q_0 - p_0|\right) - |q_0 - p_0|| \tag{3.10}$$

for all $t \in \mathbb{N}_0$.

Now, we claim that $\psi(y) \leq Ly + \epsilon$, $\forall y \geq 0$. To prove this, we claim by using contradiction method, let us assume the contrary that there is some $y_0 \geq 0$ such that $\psi(y_0) > Ly_0 + \epsilon$. Then we have

$$|\rho(y_0) - \rho(0)| = |\psi(y_0) - \psi(0)| > L\,|y_0 - 0|\,.$$

This gives that $|\rho(y_0) - \rho(0)| > L\,|y_0 - 0|$. This is contradiction to (3.5). Hence there exists a unique fixed point y^* such that $y^* = \psi(y^*) \leq Ly^* + \epsilon$. This gives

$$y^* \leq \frac{\epsilon}{1-L} \tag{3.11}$$

Using (3.11) in (3.10), we get

$$|q_{t+n-1} - p_{t+n-1}| \leq \frac{\epsilon}{1-L} + \frac{L^{t+n-1}}{1-L}\,|\psi\left(|q_0 - p_0|\right) - |q_0 - p_0|| \tag{3.12}$$

for all $t \in \mathbb{N}_0$. □

Now, the ensuing theorem shows that the Hyers-Ulam stability of the nth order difference equation (1.1) under some more explicit condition for the complex-valued function g and an additional condition that $p_0 = q_0$.

Corollary 3.1. *Given real constants ϵ and L with $0 < L < 1$, let $g : \mathbb{N}_0 \times \mathbb{C}^n \to \mathbb{C}^n$ be a function fulfilling the condition*

$$|g(t, v) - g(t, w)| \leq L\,|v - w|$$

for all $t \in \mathbb{N}_0$ and $v, w \in \mathbb{C}^n$. If a complex-valued sequence $\{p_t\}_{t\in\mathbb{N}_0}$ such that

$$|p_{t+n} - g\left(t, p_t, \ldots, p_{t+n-1}\right)| \leq \epsilon$$

for all $t \in \mathbb{N}_0$. Then there exists a complex-valued sequence $\{q_t\}_{t\in\mathbb{N}_0}$ such that

$$q_{t+n} = g\left(t, q_t, \ldots, q_{t+n-1}\right)$$

and

$$|q_{t+n-1} - p_{t+n-1}| \leq \frac{1}{1-L}\left[\epsilon\left(1 + L^{t+n-1}\right) + L^{t+n-1}\,|q_0 - p_0|\right]$$

for all $t \in \mathbb{N}_0$,

Proof. First, let us define a monotonically increasing contraction mappings $\rho, \psi : [0, \infty) \to [0, \infty)$ defined by $\rho(y) = L(y)$ and $\psi(y) = \rho(y) + \epsilon = Ly + \epsilon$, then by using Theorem 3.2, we have

$$\begin{aligned}|q_{t+n-1} - p_{t+n-1}| &\leq \frac{\epsilon}{1-L} + \frac{L^{t+n-1}}{1-L}\,|\psi\left(|q_0 - p_0|\right) - |q_0 - p_0|| \\ &= \frac{1}{1-L}\left[\epsilon\left(1 + L^{t+n-1}\right) + L^{t+n-1}\,|q_0 - p_0|\right]\end{aligned}$$

for all $t \in \mathbb{N}_0$. □

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