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SOLUTION OF NONLINEAR FOURTH ORDER BOUNDARY VALUE PROBLEMS BY SHOOTING TYPE DIFFERENTIAL TRANSFORM ALGORITHM

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ABSTRACT. In this paper, Shooting Type Differential Transform Algorithm (STDTA) has been used to solve some nonlinear fourth order boundary value problems. Using STDTA, the problems are solved and the solution is calculated in the form of a rapid convergent series. It demonstrates the efficiency and simplicity of the proposed method.

1. INTRODUCTION

The differential transform method (DTM) is an analytical method based on the Taylor expansion. This method gives an analytical solution in the form of a polynomial. The concept of differential transform method was first proposed and applied to solve linear and nonlinear initial value problem in electric circuit analysis by Zhou [1]. The differential transform method is an iterative procedure that is described by the transformed equations of original functions for solution of differential equations. In this paper, three nonlinear fourth order boundary value problems are solved using shooting type differential transform algorithm. In Section 2, we give some basic properties of one-dimensional DTM and explain the procedure of STDTA. In Section 3, we have applied the method to solve nonlinear boundary value problems.

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2. One-Dimensional Differential Transform Method and their properties

In this section, we first give some basic properties [2] of one-dimensional differential transform method. Differential transform of a function $y(x)$ is defined as follows

$$Y(k) = \frac{1}{k!}\frac{d^k(y)}{dx^k}, \tag{2.1}$$

where $y(x)$ is the original function and $Y(k)$ is the transformed function for $k = 0, 1, 2, \ldots$. The differential inverse transform of $Y(k)$ is defined as

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k. \tag{2.2}$$

From equations (2.1) and (2.2) we get

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\frac{d^k y}{dx^k}. \tag{2.3}$$

Theorem 2.1. *If* $f(x) = g(x) \pm h(x),$ *then* $F(k) = G(k) \pm H(k).$

Theorem 2.2. *If* $f(x) = \lambda g(x),$ *then* $F(k) = \lambda G(k)$ *where* λ *is a constant.*

Theorem 2.3. *If* $f(x) = g(x)h(x),$ *then* $F(k) = \sum_{r=0}^{k} G(r)H(k-r).$

Theorem 2.4. *If* $f(x) = g(x)\dfrac{dg(x)}{dx},$ *then* $F(k) = \sum_{r=0}^{k}(k-r+1)G(r)G(k-r+1).$

Theorem 2.5. *If* $f(x) = x^m,$ *then* $F(k) = \delta(k-m)$ *where* $\delta(k-m) = \begin{cases} 1, & k = m \\ 0, & k \neq m. \end{cases}$

Let B be a Banach space and consider the functional equation defined on the Banach space B, $Ty = b$ where T is an operator from B to B, b is a given function of B, and for each satisfying the functional equation [3, 4] is the solution. Assume that the functional equation has a unique solution for each $b \in B$.

The operator T consists of both linear and nonlinear terms, the linear term is decomposed into $L_1 + L_2$, where L_1 is the invertible, highest order derivative and L_2 is the remainder of the linear operator. Thus $T = L_1 + L_2 + N$ where N is a nonlinear operator. Hence the functional equation becomes

$$L_1 y = b - L_2 y - Ny.$$

Taking the Differential Transform on both sides of the above equation, we get the transformed equation as

$$Y(k+n) = \frac{F(k)}{(k+n)!}, \tag{2.4}$$

where $F(k)$ is the differential transform of

$$f(x, y, y', y'', \ldots, y^{(n-1)}) = b - L_2 y - Ny.$$

Then transformed conditions given with the problem can be written as

$$Y(k) = J, \quad Y(m) = \sum_{k=0}^{N} \prod_{i=1}^{m-1} (k-i) Y(k) = I_m, (m < n), \tag{2.5}$$

where m is the order of the derivative in the boundary conditions and J, I_m are real constants. Using equations (2.4) and (2.5) the values of $Y(i), i = 1, 2, 3, \ldots$ can determined and then using inverse differential transformation, the following approximate solution can be determined as

$$Y_N = \sum_{k=0}^{N} Y(k) x^k. \tag{2.6}$$

Usually DTM is used for solving initial value problems. To solve boundary value problems efficiently the authors [5, 6] have introduced Shooting Type Differential Transform Algorithm (STDTA). The basic steps of STDTA are as follows:

(i) Converting the given boundary value problem into an initial value problem by assuming the missing initial conditions;
(If the differential equation is of order n and there are m conditions given at the initial point and the remaining $n-m$ conditions are given at other points, assumptions are made on the remaining $n-m$ initial conditions. In the case of fourth order boundary value problem one assumes $u''(0) = \alpha, u'''(0) = \beta$.)

(ii) applying the DTM to the converted initial value problem;
(In the case of fourth order boundary value problem the assumed condition transforms to $U(2) = \frac{\alpha}{2!}, U(3) = \frac{\beta}{3!}$.)

(iii) computing the coefficients $Y(k+n)$ for $k \geq 0$ using (2.4) up to a specified level; and

(iv) finding the value(s) of the assumed condition(s) by applying the boundary condition(s) at the other point to the approximate solution (2.6).

(In the case of fourth order boundary value problem α, β the two assumed constants are found out by applying the condition at the second point to the approximate solution).

The effectiveness of STDTA is demonstrated here by applying it to some fourth order nonlinear boundary value problems.

3. Illustrative Examples

Example 1. *Consider the nonlinear boundary value problem of fourth order [7]:*

$$u^{(4)}(x) = u^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48 \tag{3.1}$$

with boundary conditions

$$u(0) = u^{'}(0) = 0, \quad u(1) = 1, \quad u^{'}(1) = 1. \tag{3.2}$$

The Exact solution is $u(x) = x^5 - 2x^4 + 2x^2$. Taking the differential transform of (3.1), yields

$$\begin{aligned} U(k+4) = &\frac{1}{(k+1)(k+2)(k+3)(k+4)}\big[\sum_{r=0}^{k} U(r)U(k-r) - \delta(k-10) \\ &+ 4\delta(k-9) - 4\delta(k-8) - 4\delta(k-7) + 8\delta(k-6) \\ &- 4\delta(k-4) + 120\delta(k-1) - 48\delta(k-0)\big]. \end{aligned}$$

In the modified approach, one assumes that $u^{''}(0) = \alpha, u^{'''}(0) = \beta$. Hence $U(0) = 0$, $U(1) = 0$, $U(2) = \frac{\alpha}{2!}$, $U(3) = \frac{\beta}{3!}$. Putting $k = 0, 1, 2, 3, \ldots,$ in the transformed equation, the series coefficients $U(4), U(5), U(6), U(7), U(8), U(9), \ldots,$ can be obtained as $U(4) = -2$, $U(5) = 1$, $U(6) = 0$, $U(7) = 0$, $U(8) = \frac{(\alpha^2-16)}{47040}$, $U(9) = \frac{\alpha\beta}{18144}$, and so on. Then the successive approximations to the solutions are obtained, using $u_n(x) = \sum_{k=0}^{n} U(k)x^k$. The ninth approximation is $u_9(x) = \frac{\alpha}{2}x^2 + \frac{\beta}{6}x^3 - 2x^4 + x^5 + \frac{(\alpha^2-16)}{47040}x^8 + \frac{\alpha\beta}{18144}x^9$. The i^{th} approximation to the solution $u(x), u_i(x)$ is the terms up to x^i of the above expression.

Now applying the condition $u(1) = 1$ and $u^{'}(1) = 1$ to $u_n(x)$, the approximate values for α and β, namely α_n and β_n for different values of n, are obtained. They are tabulated in Table 1. Since $U(6)$ and $U(7)$ are zero for $n = 6, 7$ there will be no changes in $u_6(x)$ and $u_7(x)$. From the table it is clear that the sequence α_n and β_n converges. Substituting these values of α_n, β_n in the corresponding

TABLE 1. Values of α_n and β_n

n	α_n	β_n
5	4	0
6	4	0
7	4	0
8	4	0
9	4	0

TABLE 2. Comparison with the existing results

x	DTM	VIM	Exact Solution
0.1	0.0198100	0.0198099	0.0198100
0.2	0.0771200	0.0771199	0.0771200
0.3	0.1662300	0.1662299	0.1662300
0.4	0.2790400	0.2790399	0.2790400
0.5	0.4062500	0.4062499	0.4062500
0.6	0.5385600	0.5385599	0.5385599
0.7	0.6678700	0.6678699	0.6678700
0.8	0.7884800	0.7884799	0.7884800
0.9	0.8982900	0.8982899	0.8982900
1.0	1.0000000	0.9999999	1.0000000

$u_n(x)$, the n^{th} approximation to the solution, $u(x)$ is obtained. Table 2 gives the values of $u_n(x)$, evaluated at $x = 0.1, 0.2, 0.3, 0.4, \ldots, 0.9, 1.0,$ for different values of n are compared with the variational iteration method [7] and the Exact solution. From the table we see that DTM is a better approximation than VIM for this problem.

Example 2. *Consider the nonlinear beam equation [8]*

$$y^{(4)}(x) = c(y(x))^2 + 1, \qquad 0 \leq x \leq 2, \tag{3.3}$$

subject to boundary conditions,

$$y(0) = y'(0) = y(2) = y'(2) = 0. \tag{3.4}$$

TABLE 3. Values of α_n and β_n

n	α_n	β_n
4	0.33333333	-1
5	0.33333333	-1
6	0.33333333	-1
7	0.33333333	-1
8	0.34464578	-1.02036264

Let $c = 1$. Taking differential transform of (3.3), yields

$$Y(k+4) = \frac{1}{(k+1)(k+2)(k+3)(k+4)}\Big[\sum_{r=0}^{k} Y(r)Y(k-r) + \delta(k-0)\Big].$$

Proceeding as in Example 1, the series coefficients $Y(4), Y(5), Y(6), \ldots,$ can be obtained as

$$Y(4) = \frac{1}{24}, \quad Y(5) = 0, \quad Y(6) = 0, \quad Y(7) = 0, \quad Y(8) = \frac{\alpha^2}{6720},$$

and so on.

Then the successive approximations to the solution are obtained, using $y_n(x) = \sum_{k=0}^{n} Y(k)x^k$. The 8^{th} approximation is

$$y_8(x) = \frac{\alpha}{2}x^2 + \frac{\beta}{6}x^3 + \frac{1}{24}x^4 + \frac{\alpha^2}{6720}x^8.$$

The i^{th} approximation to the solution $y(x), y_i(x)$ is the terms up to x^i of the above expression. Now applying the condition $y(2) = y^{'}(2) = 0$ to $y_n(x)$, the approximate values for α and β, namely α_n and β_n, for different values of n, are obtained. They are tabulated in Table 3. From this table it is clear that the sequence α_n and β_n converges.

Substituting these values of α_n, β_n in the corresponding $y_n(x)$, the n^{th} approximation to the solution, $y(x)$ is obtained. Table 4 gives the values of $y_n(x)$, evaluated at $x = 0.0, 0.1, 0.2, 0.3, \ldots, 2.0$, for different values of n. $y_4 = \frac{1}{6}x^2(1-\frac{x}{2})^2 = \frac{1}{24}x^2(2-x)^2$.

Also $y_4 = y_5 = y_6 = y_7$. They are symmetric about the line $x = 1$. But we see that y_8 looses the symmetry. Table 5 gives the comparison with the

TABLE 4. Convergence of y_n

x	$y_4 = y_5 = y_6 = y_7$	y_8
0.0, 2.0	0.00000000	0.00000000
0.1, 1.9	0.00150417	0.00155734, 0.00164721
0.2, 1.8	0.00539999	0.00539999, 0.00588153
0.3, 1.7	0.01083749	0.01125493, 0.01174339
0.4, 1.6	0.01706666	0.01775447, 0.01840487
0.5, 1.5	0.02343749	0.02485162, 0.02516303
0.6, 1.4	0.02939999	0.03070349, 0.03143454
0.7, 1.3	0.03450417	0.03611267, 0.03675125
0.8, 1.2	0.03839999	0.04028536, 0.04075652
0.9, 1.1	0.04083749	0.04295259, 0.04320231
1.0	0.04166666	0.04394679

existing results: Iteration of the integral equation (IIE), Trapezoidal rule(TRAP), Simpson rule (SIMP), Classical Adomian method (ADM) [9].

Example 3. *Consider the nonlinear boundary value problem [10, 11]*

(3.5) $$v^{(4)} - v^2 = \mathrm{e}^t - t^4 - \mathrm{e}^{2t} - 2t^2\mathrm{e}^t, \qquad 0 < t < 1,$$

subject to

(3.6) $$v(0) = 1, \qquad v(1) = 1 + \mathrm{e}, \qquad v^{'}(0) - v(0) = 0, \qquad v^{'}(1) - v(1) = 1.$$

The Exact solution is $v(t) = t^2 + \mathrm{e}^t$.

Taking the differential transform of (3.5), yields

$$\begin{aligned} V(k+4) &= \frac{1}{(k+1)(k+2)(k+3)(k+4)}\Big[\sum_{r=0}^{k} V(r)V(k-r) + \frac{1}{k!} - \delta(k-4) \\ &\quad -\frac{2^k}{k!} - 2\sum_{r=0}^{k}\frac{\delta(r-2)}{(k-r)!}\Big]. \end{aligned}$$

Proceeding as in Example 1, the series coefficients $V(4), V(5), V(6), \ldots,$ can be obtained as $V(4) = \frac{1}{24}$, $V(5) = \frac{1}{120}$, $V(6) = \frac{1}{360}[\alpha - \frac{5}{2}]$, $V(7) = \frac{1}{840}[\alpha + \frac{\beta}{3} - \frac{19}{6}]$, $V(8) = \frac{1}{40320}[6\alpha^2 + 8\beta - 61]$, $V(9) = \frac{1}{3024}[\frac{\alpha\beta}{6} - \frac{59}{120}]$, and so on.

The 9^{th} approximation is $v_9(t) = 1 + t + \frac{\alpha}{2}t^2 + \frac{\beta}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + \frac{1}{360}[\alpha - \frac{5}{2}]t^6 + \frac{1}{840}[\alpha + \frac{\beta}{3} - \frac{19}{6}]t^7 + \frac{1}{40320}[6\alpha^2 + 8\beta - 61]t^8 + \frac{1}{3024}[\alpha\frac{\beta}{6} - \frac{59}{120}]t^9$.

TABLE 5. Comparison with the existing results

x	DTM	IIE	TRAP	SIMP	ADM
0.0, 2.0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
0.1, 1.9	0.00150417	0.00150566	0.00150566	0.00150566	0.00150565
0.2, 1.8	0.00539999	0.00540549	0.00540548	0.00540549	0.00540547
0.3, 1.7	0.01083749	0.01084878	0.01084878	0.01084878	0.01084876
0.4, 1.6	0.01706666	0.01708484	0.01708483	0.01708484	0.01708480
0.5, 1.5	0.02343749	0.02346298	0.02346295	0.02346298	0.02346292
0.6, 1.4	0.02939999	0.02943253	0.02943250	0.02943253	0.02943246
0.7, 1.3	0.03450417	0.03454290	0.03454287	0.03454290	0.03454281
0.8, 1.2	0.03839999	0.03844357	0.03844355	0.03844357	0.03844347
0.9, 1.1	0.04083749	0.04088413	0.04088413	0.04088413	0.04088403
1.0	0.04166666	0.04171435	0.04171435	0.04171435	0.04171425

TABLE 6. Values of α_n and β_n

n	α_n	β_n
7	2.99971033	1.00104058
8	2.99996192	1.00013317
9	2.99999566	1.00001491

The i^{th} approximation to the solution $v(t), v_i(t)$ is the terms up to t^i of the above expression.

Now applying the condition, $v(1) = 1 + \text{e}$ and $v^{'}(1) = 2 + \text{e}$ to $v_n(t)$, the approximate values for α and β, namely α_n and β_n, for different values of n, are obtained. They are tabulated in Table 6. From this table it is clear that the sequence α_n and β_n converges. Substituting these values of α_n, β_n in the corresponding $v_n(t)$, the n^{th} approximation to the solution, $v(t)$is obtained. Table 7 gives the values of $v_n(t)$, evaluated at $t = 0.1, 0.2, 0.3, 0.4, \ldots, 0.9, 1.0,$ for different values of n.

TABLE 7. Convergence of v_n

x	v_7	v_8	v_9	Exact Solution
0.1	1.11517065	1.11517083	1.11517089	1.11517092
0.2	1.26140644	1.261402494	1.26140269	1.26140276
0.3	1.43987773	1.439858413	1.43985868	1.43985881
0.4	1.65187727	1.651824351	1.65182451	1.65182469
0.5	1.89883294	1.89872128	1.89872104	1.89872127
0.6	2.18232194	2.18211959	2.18211855	2.18211880
0.7	2.50408633	2.50375478	2.50375248	2.50375271
0.8	2.86605007	2.86554480	2.86554078	2.86554093
0.9	3.27033767	3.26960912	3.26960305	3.26960311
1.0	3.71929458	3.71828982	3.71828183	3.71828183

4. CONCLUSION

In this work, Shooting Type Differential Transform Algorithm has been successfully applied to solve nonlinear boundary value problems. The three examples solved revealed that the method is fast, accurate and easy to apply.

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