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ON FUZZY TOPOLOGICAL *BRK*-SUBALGEBRAS

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ABSTRACT. In this paper, fuzzy topological BRK-subalgebras of a BRK-algebras is introduced. Also, fuzzy topological BRK-ideals in BRK-algebras is also introduced and discussed some of their properties.

1. INTRODUCTION

Imai and Iseki [4] subjected two classes of abstract algebras: BCK-algebras and BCI-algebras in the year of 1996. In 1983, the notion of a BCH-algebra was introduced by Hu and Li [3], which is a generalization of BCK and BCI-algebras. In 2002, a new notion B-algebra was introduced by Neggers and Kim [9]. Also a BF-algebra and BG-algebra was introduced by Walendziak [13] in 2007 and C. B. Kim and H. S. Kim [6], which is a generalization of B-algebra. The concept of a fuzzy set was introduced in [14], provides a general topology called fuzzy topological spaces. The structure of a fuzzy topological spaces by D. H. Foster [2] combined with a fuzzy group. A. Rosenfeld [11] has formulate the elements of a theory of fuzzy topological groups. In 2012, R. K. Bandaru [10] introduced BRK-algebra, which is a generalization of $BCK/BCI/BCH/Q/QS/BM$-algebras [5,7,8]. In [1], El-Gendy introduced the notion of fuzzy BRK-ideal of BRK-algebra. S. Sivakumar et al. introduced a topology on BRK-algebra [12] and also studied several concepts. In the present paper, $f\tau BRK\ SA$ is introduced in a BRK-algebras.

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2. PRELIMINARIES

Definition 2.1. *[10] A BRK-algebra (briefly, BRK Alg) $(I, \star, 0)$ is a non-empty set I with a constant 0 and a binary operation $\star$ satisfying the following axioms:*

$$(BRK_1)\ i_1 \star 0 = i_1,$$

$$(BRK_2)\ (i_1 \star i_2) \star i_1 = 0 \star i_2$$

for any $i_1, i_2 \in I$. In a BRK Alg I, $\leq$ a partially ordered relation can be defined by $i_1 \leq i_2$ iff $i_1 \star i_2 = 0$.

Definition 2.2. *[10] Let I be a BRK Alg and $a \in I$. Define a right map $R_a : I \to I$ by $R_a(i) = i \star a\ \forall\ i \in I$.*

Definition 2.3. *[12] Let $(I, \star, 0)$ be a BRK Alg and τ a topology on I. Then $I = (I, \star, 0, \tau)$ is called a topological BRK Alg (briefly, τBRK Alg), if "$\star$" is continuous or equivalently, for any $m, n \in X$ and $\forall$ O open set of $m \star n$, $\exists$ two open sets $M \& N$ respectively, such that $M \star N$ is a subset of O.*

Definition 2.4. *[12] Let a non-empty subset M of a τBRK Alg I, then M is called τBRK-subalgebra (briefly, τBRK sub Alg) of I if*

$$i_1 \star i_2 \in M\ \forall\ i_1, i_2 \in M.$$

Definition 2.5. *[12] Let I be a τBRK Alg and D be a subset of I, then D is called a τBRK-ideal (briefly, τBRK I) of I, if for any $i_{11}, i_{22} \in I$:*

(i) $0 \in D$,

(ii) $0 \star (i_{11} \star i_{22}) \in D$ *and* $0 \star i_{22} \in D$ *imply* $i_{11}, i_{22} \in I$.

Definition 2.6. *[1] Let I be a set. A function $\mu_I : I \to [0, 1]$ where μ_I a fuzzy set in I.*

Definition 2.7. *[2] A fuzzy topology (briefly, ft) on a set I is a family τ of fuzzy subsets in I satisfies*

(i) $\forall$ $c \in [0, 1]$, $K_c \in \tau$, *where K_c have constant membership functions with the value c,*

(ii) *If $K, L \in \tau$, then $K \cap L \in \tau$,*

(iii) *If $K_j \in \tau\ \forall\ j \in J$, then $\bigcup_{j \in J} K_j \in \tau$.*

The pair (I, τ) is called a fuzzy topological space (briefly, fts) and members of τ are fuzzy open (briefly, τfo) subsets.

Definition 2.8. *[2] Let M be a fuzzy subset in I and τ a ft on I. Then the induced ft on M is the family of fuzzy subsets of M which are the intersection with M of τfo subsets in I. The induced ft is denoted by τ_M and the pair (M, τ_M) is called a fuzzy subspace of (I, τ).*

3. Fuzzy topological BRK-subalgebras

Definition 3.1. *The pair (I, τ) is called a fts, then it satisfies a $BRK\ Alg$ properties in $(I, \star, 0, \tau)$ it is called a fuzzy BRK topological spaces (briefly, $fBRKts$) and members of τ are BRK-open fuzzy (briefly, $BRK\ of$) sets.*

Definition 3.2. *A fuzzy subset K in a $BRK\ Alg$, $(I, \star, 0)$ with membership function μ_K is called a fuzzy BRK-subalgebra (briefly, $fBRK\ SA$) of I if*

$$(\forall i_1, i_2 \in I)[\mu_K(0 \star (i_1 \star i_2)) \geq \min\{\mu_K(0 \star i_1), \mu_K(0 \star i_2)\}].$$

Example 1. *Let $(I = \{0, a_{a_1}, a_{a_2}, a_{a_3}\}, \star, 0)$ be a $BRK\ Alg$ defined by*

$\star$	0	a_1	a_2	a_3
0	0	0	a_2	a_2
a_1	a_1	0	a_2	a_2
a_2	a_2	a_2	0	0
a_3	a_3	a_3	a_1	0

A fuzzy subset K in I defined by $\mu_K(a_3) = 0.4$ and $\mu_K(a_x) = 0.8$ for all $a_x \neq a_3$ is a $fBRK\ SA$ of I.

Definition 3.3. *Let (I, τ) and (J, τ') be two fts's. A mapping ψ of (I, τ) into (J, τ') is fuzzy BRK continuous (briefly, $fBRK\ Cts$) if for each fos U in τ', the inverse image $\psi^{-1}(U)$ is $fBRKo$ in τ. Conversely, ψ is fuzzy BRK open (briefly, $fBRKO$) if for each fos V in τ, the image $\psi(V)$ is $fBRKo$ in τ'.*

Definition 3.4. *Let (A, τ_A) and (B, τ'_B) be fuzzy subspaces of fts's (I, τ) and (J, τ') respectively and let a map $\psi : (I, \tau) \to (J, \tau')$. Then a mapping ψ of (A, τ_A) into (B, τ'_B) if $\psi(A) \subset B$. Furthermore ψ is relatively fuzzy BRK continuous (briefly, $rfBRK\ Cts$) if for each fos V' in τ'_B, the intersection $\psi^{-1}(V') \cap A$ is $fBRKo$ in τ_A. Conversely, ψ is relatively fuzzy BRK open (briefly, $rfBRKO$) if for each fos U' in τ_A, the image $\psi(U')$ is $fBRKo$ in τ'_B.*

Proposition 3.1. *Let $(A, \tau_A), (B, \tau'_B)$ be fuzzy subspaces of $fBRKts$'s I and J respectively. Let ψ be a $fBRK\ Cts$ mapping of (I, τ) into (J, τ') such that $\psi(A) \subset B$. Then ψ is a $rfBRK\ Cts$ mapping of (A, τ_A) into (B, τ'_B).*

Proposition 3.2. *Let ψ be a homomorphism of a $BRK\ Alg$'s I into J and S a $fBRK\ SA$ of J with membership function μ_S. Then the inverse image $\psi^{-1}(S)$ of S is a $fBRK\ SA$ of I.*

Proof. Let $i_1, i_2 \in I$. Then

$$\begin{aligned}
\mu_{\psi^{-1}(S)}(0 \star (i_1 \star i_2)) &= \mu_S(\psi(0 \star (i_1 \star i_2))) = \mu_S(\psi(0 \star i_1)\psi(0 \star i_2)) \\
&\geq \min\{\mu_S(\psi(0 \star i_1)), \mu_S(\psi(0 \star i_2))\} \\
&= \min\{\mu_{\psi^{-1}(S)}(0 \star i_1), \mu_{\psi^{-1}(S)}(0 \star i_2)\}.
\end{aligned}$$

This completes the proof. □

Definition 3.5. *A fuzzy subset K in a $BRK\ Alg$ I with μ_K is said to be fuzzy BRK-sup property (briefly, $fBRK\ SP$) if for any subset $P \subset I$, there exists $p_0 \in P$ such that*

$$\mu_K(0 \star p_0) = \sup_{p \in P} \mu_K(0 \star p). \tag{3.1}$$

Proposition 3.3. *Let ψ be a homomorphism of a $BRK\ Alg$'s I onto J and let S be a $fBRK\ SA$ of I that has the $fBRK\ SP$. Then the image $\psi(K)$ of K is a $fBRK\ SA$ of J.*

Proof. For $j_1, j_2 \in J$, let $x_0 \in \psi^{-1}(j_1)$; $y_0 \in \psi^{-1}(j_2)$ such that

$$\mu_K(0 \star x_0) = \sup_{t \in \psi^{-1}(j_1)} \mu_K(0 \star t),$$

$$\mu_K(0 \star y_0) = \sup_{t \in \psi^{-1}(j_2)} \mu_K(0 \star t).$$

Then, by $\mu_\psi(0 \star K)$,

$$\begin{aligned}
\mu_{\psi(K)}(0 \star (j_1 \star j_2)) &= \sup t \in \psi^{-1}(j_1 j_2) \mu_K(0 \star t) \geq \mu_K(0 \star (x_0 \star y_0)) \\
&\geq \min\{\mu_K(0 \star x_0), \mu_K(0 \star y_0)\} \\
&= \min\{\sup_{t \in \psi^{-1}(j_1)} \mu_K(0 \star t), \sup_{t \in \psi^{-1}(j_2)} \mu_K(0 \star t)\} \\
&= \min\{\mu_{\psi(K)}(0 \star j_1), \mu_{\psi(K)}(0 \star j_2)\},
\end{aligned}$$

ending the proof. □

Definition 3.6. *Let I be a $BRK\ Alg$ and τ a ft on I. Let S be a $fBRK\ SA$ of I with induced topology τ_S. Then S is called a fuzzy topological BRK-subalgebra (briefly, $f\tau BRK\ SA$) of I if $\forall\ a \in I$ the mapping $R_a : (0 \star i) \to (i \star a)$ of $(S, \tau_S) \to (S, \tau_S)$ is $rfBRK\ Cts$.*

Theorem 3.1. *Let I, J be a $BRK\ Alg$'s and a homomorphism $\psi : I \to J$. let τ and τ' be ft's on I & J respectively, such that $\tau = \psi^{-1}(\tau')$. Let S be a $f\tau BRK\ SA$ of J with μ_S. Then $\psi^{-1}(S)$ is a $f\tau BRK\ SA$ of I with $\mu_{\psi^{-1}(S)}$.*

Proof. To show that, $\forall\ a \in I$, the mapping

$$R_a : (0 \star i) \to (i \star a) \text{ of } (\psi^{-1}(S), \tau_{\psi^{-1}}(S)) \to (\psi^{-1}(S), T_{\psi^{-1}}(S)) \tag{3.2}$$

is $rfBRK\ Cts$. Let U_0 be an $fBRKo$ set in $\tau_{\psi^{-1}}(S)$ on $\psi^{-1}(S)$. Since ψ is a $fBRK\ Cts$ mapping of (I, τ) into (J, τ'), it follows from Proposition 3.1 that ψ is a $rfBRK\ Cts$ mapping of $(\psi^{-1}(S), \tau_{\psi^{-1}}(S))$ into (S, τ'_S).

Note that $\exists\ fos\ V_0 \in \tau'_S \ni \psi^{-1}(V_0) = U_0$. The membership function of $R_a^{-1}(U_0)$ is given by

$$\begin{aligned} \mu_{R_a{}^{-1}(U_0)}(0 \star i) &= \mu_{U_0}(R_a(0 \star i)) = \mu_{U_0}(i \star a) = \mu_{\psi^{-1}(V_0)}(i \star a) \\ &= \mu_{V_0}(\psi(i \star a)) = \mu_{V_0}(\psi(0 \star i)\psi(0 \star a)). \end{aligned} \tag{3.3}$$

As S is a $f\tau BRK\ SA$ of J, the mapping

$$R_b : (0 \star j) \to (j \star b) \text{ of } (S, \tau'_S) \to (S, \tau'_S) \tag{3.4}$$

is $rfBRK\ Cts$ for each $b \in J$. Hence

$$\begin{aligned} \mu_{R_a{}^{-1}(U_0)}(0 \star i) &= \mu_{V_0}(\psi(0 \star i)\psi(0 \star R_a)) = \mu_{V_0}(\psi(0 \star R_a)(\psi(0 \star i))) \\ &= \mu_{\psi(0 \star R_a)^{-1}(V_0)}(\psi(0 \star i)) = \mu_{\psi^{-1}(\psi(R_a)^{-1}(V_0))}(0 \star i), \end{aligned}$$

which implies that $R_a^{-1}(U_0) = \psi^{-1}(\psi(a)_r^{-1}(V_0))$ so that

$$R_a^{-1}(U_0) \cap \psi^{-1}(S) = \psi^{-1}(\psi(R_a)^{-1}(V_0)) \cap \psi^{-1}(S) \tag{3.5}$$

is open in the induced ft on $\psi^{-1}(S)$. The proof is complete. □

We say that μ_S of a $fBRK\ SA\ S$ of a $BRK\ Alg\ I$ is ψ-invariant [11] if $\forall\ i_1, i_2 \in I$, $\psi(i_1) = \psi(i_2)$ implies $\mu_S(0 \star i_1) = \mu_S(0 \star i_2)$. Clearly, $\psi(S)$ of S is a homomorphic image then a $fBRK\ SA$.

Theorem 3.2. *Let I, J be a BRK Alg's and a homomorphism ψ of I onto J, let τ and τ' be a ft's on I & $J \ni \psi(\tau) = \tau'$. Let S be a $f\tau BRK$ SA of I. If μ_S of S is ψ-invariant, then $\psi(S)$ is a $f\tau BRK$ SA of J.*

Proof. It is sufficient that,

$$R_b : (0 \star j) \to (j \star b) \text{ of } (\psi(S), \tau'_{\psi(S)}) \to (\psi(S), \tau'_{\psi(S)}) \tag{3.6}$$

is $rfBRK$ Cts for each $b \in J$. Note that ψ is $rfBRKO$ for if $U_0{}' \in \tau_S$, there exists $U_0 \in \tau \ni U_0{}' = U_0 \cap S$ and by the ψ-invariance of μ_S,

$$\psi(U_0{}') = \psi(U_0) \cap \psi(S) \in \tau'_{\psi(S)}. \tag{3.7}$$

Let $V_0{}'$ be a fos in $\tau'_{\psi(S)}$. Since ψ is onto, $\forall$ $b \in J \exists$ $a \in I$ such that $b = \psi(a)$. Hence

$$\begin{aligned}
\mu_{\psi^{-1}(R_b^{-1}(V_0{}'))}(0 \star i) &= \mu_{\psi^{-1}(\psi(R_a)^{-1}(V_0{}'))}(0 \star i) = \mu_{\psi(R_a)^{-1}(V_0{}')}(\psi(0 \star i)) \\
&= \mu'_{V_0}(\psi(0 \star R_a)(\psi(0 \star i))) = \mu'_{V_0}(\psi(0 \star i)\psi(0 \star R_a)) \\
&= \mu'_{V_0}(\psi(0 \star (i \star R_a))) = \mu_{\psi^{-1}(V_0{}')}(0 \star (i \star R_a)) \\
&= \mu_{\psi^{-1}(V_0{}')}(R_a \star (0 \star i)) = \mu_{R_a^{-1}(\psi^{-1}(V_0{}'))}(0 \star i),
\end{aligned}$$

which implies that $\psi^{-1}(R_b^{-1}(V_0{}')) = R_a^{-1}(\psi^{-1}(V_0{}'))$.

By hypothesis, $R_a : (0 \star i) \to (0 \star (i \star a))$ is a $rfBRK$ Cts mapping $(S, \tau_S) \to (S, \tau_S)$ and ψ is a $rfBRK$ Cts mapping $(S, \tau_S) \to (\psi(S), \tau'_{\psi(S)})$. Hence

$$\psi^{-1}(R_b^{-1}(V_0{}')) \cap T = R_a^{-1}(\psi^{-1}(V_0{}')) \cap S \tag{3.8}$$

is open in τ_S. Since ψ is $rfBRKO$,

$$\psi(\psi^{-1}(R_b^{-1}(V_0{}')) \cap S) = R_b^{-1}(V_0{}') \cap \psi(S) \tag{3.9}$$

is $fBRKo$ in $\tau'_{\psi(S)}$. The proof is complete. □

4. Fuzzy topological BRK-ideals

Definition 4.1. *A fuzzy subset K in I with μ_K is called a fuzzy BRK-ideal (briefly, $fBRKI$) of I if*

(i) ($\forall i_1 \in I$), $[\mu_K(0 \star 0) \geq \mu_K(0 \star i_1)]$,
(ii) ($\forall i_1, i_2 \in I$), $[\mu_K(0 \star i_1) \geq \min\{\mu_K(0 \star (i_1 \star i_2)), (\mu_K(0 \star i_2))\}]$.

Proposition 4.1. *Let ψ be a homomorphism of a BRK Alg's I into J and L is a $fBRKI$ of J with μ_L. Then the inverse image $\psi^{-1}(L)$ of L is a $fBRKI$ of I.*

Proof. Since ψ is a homomorphism of $(I, \star, 0)$ into $(J, \star', 0')$, then $\psi(0) = 0'$. By assumption,

$$\mu_L(\psi(0 \star 0)) = \mu_L(0 \star 0') \geq \mu_L(0 \star j), \ \forall \ j \in J. \tag{4.1}$$

In particular, $\mu_L(\psi(0 \star 0)) \geq \mu_L(\psi(0 \star i)) \ \forall \ i \in I$. Thus

$$\mu_{\psi^{-1}(L)}(0 \star 0) \geq \mu_{\psi^{-1}(L)}(0 \star i), \tag{4.2}$$

which proves (i).

Now, let $i_1, i_2 \in I$. Then by μ_L,

$$\begin{aligned}\mu_{\psi^{-1}(L)}(0 \star i_1) = \mu_L(\psi(0 \star (i_1 \star i_2))) &= \mu_L(\psi(0 \star (i_1 \star i_2))\psi(0 \star i_2)) \\ &\geq \min\{\mu_L(0 \star (i_1 \star i_2)), \mu_L(0 \star i_2)\}\end{aligned}$$

which proves (ii). The proof is complete. □

Definition 4.2. *Let $(I, \star, 0, \tau)$ be a $\tau BRK\ Alg$. A fuzzy set μ_I in I is called an fuzzy topological BRK-ideal (briefly, $f\tau BRKI$) of I if*

$$(BRKI_1)\ \mu_I(0) \geq \mu_I(i_0), \tag{4.3}$$

$$(BRKI_2)\ \mu_I(0 \star i_1) \geq \min\{\mu_I(0 \star (i_1 \star i_2)), \mu_I(0 \star i_2)\}, \ \forall \ i_1, i_2 \in I. \tag{4.4}$$

Since any $fBRK\ I$ is a $fBRK\ SA$, then a $f\tau BRK\ I$ is a $f\tau BRK\ SA$ and from Theorem 3.1 and Proposition 4.1, we obtain a corollary 4.1.

Corollary 4.1. *Let I, J be a $BRK\ Alg$'s and a homomorphism $\psi : I \to J$, let τ and τ' be ft's on I & J respectively, such that $\tau = \psi^{-1}(\tau')$. Let H be a $f\tau BRKI$ of J with μ_H. Then $\psi^{-1}(H)$ is a $f\tau BRKI$ of I with $\mu_{\psi^{-1}}(H)$.*

If μ_H of a $fBRKI\ H$ of a $BRK\ Alg\ I$ is ψ-invariant, then $\psi(H)$ of H is a homomorphic image of a $fBRKI$. Thus Theorem 3.2 follows a Corollary 4.2.

Corollary 4.2. *Given I, J be a $BRK\ Alg$'s and a homomorphism ψ of I onto J, let τ and τ' be a ft's on I and J such that $\psi(\tau) = \tau'$. Let H be a $f\tau BRKI$ of I. If μ_H of H is ψ-invariant, then $\psi(H)$ is a $f\tau BRKI$ of J.*

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