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DECOMPOSITION OF ZERO DIVISOR GRAPH IN A COMMUTATIVE RING

A. KUPPAN AND J. RAVI SANKAR[1]

ABSTRACT. Let R be a commutative ring and let $\Gamma(Z_n)$ be the zero divisor graph of a commutative ring R, whose vertices are non-zero zero divisors of Z_n, and such that the two vertices u, v are adjacent if n divides uv. In this paper, we introduce the concept of Decomposition of Zero Divisor Graph in a commutative ring and also discuss some special cases of $\Gamma(Z_{2^2p})$, $\Gamma(Z_{3^2p})$, $\Gamma(Z_{5^2p})$, $\Gamma(Z_{7^2p})$ and $\Gamma(Z_{p^2q})$.

1. INTRODUCTION

All graphs considered here are finite and undirected, unless otherwise noted. For the standard graph-theoretic terminology the reader is referred to [4].

As usual K_n denotes the complete graph on n vertices and $K_{m,n}$ denotes the complete bipartite graph with parts of sizes m and n. Let P_k denote a path of length k and let S_k denote a star with k edges. Let C_k denotes a cycle of length K, i.e., $S_k \equiv K_{1,k}$. Let $D(K_{m,n})$ be the decomposition of complete bipartite graph. Let $L = \{H_1, H_2, ..., H_r\}$ be a family of subgraphs of G. An L-decomposition of G is an edge-disjoint decomposition of G into positive integer α_i copies of H_i where $i \in \{1, 2, 3, ..., r\}$. Furthermore, if each $H_i(i \in \{1, 2, 3, ..., r\})$ is isomorphic to a graph H, then we say that G has a H-decomposition.

The zero divisor graph is very useful to find the algebraic structures and properties of rings. The idea of a zero divisor graph of a commutative ring was

[1]*corresponding author*

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introduced by I. Beck's in [3]. Given a ring R, let G(R) denote the graph whose vertex set is R, such that distinct vertices r and s are adjacent provided that $rs = 0$. I. Beck's main interest was the chromatic number $\chi(G(R))$ of the graph G(R). The general terminology and notation are based on the papers [1,2,5–9]. In this paper we investigate the decomposition of $\Gamma(Z_{p^2q})$ into cycles and stars, and obtain the following results.

2. Preliminaries

Definition 2.1. *[1] Let R be a commutative ring (with 1) and let Z(R) be its set of zero-divisors. We associate a (simple) graph $\Gamma(R)$ to R with vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero zero-divisor of R, and for distinct $x, y \in Z(R)^*$ the vertices x and y are adjacent if and only if $xy = 0$. Thus $\Gamma(R)$ is the empty graph if and only if R is an integral domain.*

Definition 2.2. *A graph G is decomposable into $H_1, H_2, H_3, ..., H_k$ if G has subgraphs $H_1, H_2, H_3, ..., H_k$ such that*

(1) *each edge of G belongs to one of the $H_i's$ for some $i = 1, 2, 3, ..., k$ and*
(2) *If $i \neq j$, then H_i and H_j have no edges in common.*

3. 4-Partite of $\Gamma(Z_{p^2q})$

In this section, we study the decomposition of $\Gamma(Z_{p^2q})$ using cycles and stars. We first discuss the graph $\Gamma(Z_{p^2q})$ is partition into 4-partite graph.

Theorem 3.1. *If $n = 2^2p$ where p is any prime number and $p > 2$ then $\Gamma(Z_{2^2p})$ is 2-partite (bipartite) graph.*

Proof. Let $\Gamma(Z_{2^2p})$ be the non-zero zero divisor graph and let p be the prime number with $p > 2$. The vertex set of $\Gamma(Z_{2^2p})$ is $V = \{2, 4, 6, ..., 2^2p - 2, p, 2p, 3p, ..., 2^2p - p\}$. Let the vertex set can be partition into two disjoint subsets, $V_1, V_2 \in V(\Gamma(Z_{2^2p}))$ where $V_1 = \{2, 4, 6, ..., 2^2p - 2\}$ and $V_2 = \{p, 2p, 3p, ..., 2^2p - p\}$. Let $u, v \in V_1$ then 2^2p not divides uv, then does not exit the edges from u to v. Let $u \in V_1$ and $v \in V_2$ be two vertices, then 2^2p must divides uv, then there exits an edge connected between u and v. Clearly, the graph $\Gamma(Z_{2^2p})$ is 2-partite (bipartite) graph. □

Example 1. *Consider* $\Gamma(Z_{2^2 5}) = \Gamma(Z_{20})$*. The vertex set of* $\Gamma(Z_{20}) = \{2, 4, 5, 6, 8, 10, 12, 14, 15, 16, 18\}$*. The vertex set can be partition into two subsets namely* V_1 *and* $V_2 \in \Gamma(Z_{20})$ *where* $V_1 = \{5, 10, 15\}$*,* $V_2 = \{2, 4, 6, 8, 12, 14, 16, 18\}$*. Let us take any two vertices* V_1 *(or* V_2*) is non-adjacent. Theorefore Figure 1 clearly shows that the graph* $\Gamma(Z_{20})$ *is 2-partite (bipartite) graph.*

Example of a figure.

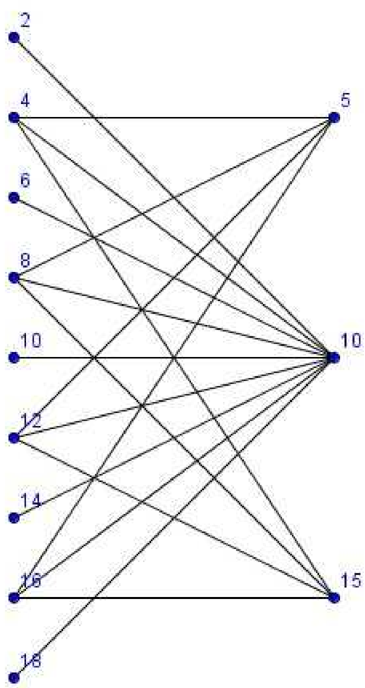

FIGURE 1. $\Gamma(Z_{20})$

Theorem 3.2. *If* p *and* q *are distinct prime numbers with* $q > p$*, then* $\Gamma(Z_{p^2 q})$ *is a 4-partite graph.*

Proof. Let p and q are distinct prime numbers and $p < q$. The vertex set of $\Gamma(Z_{p^2 q})$ is $V = \{p, 2p, 3p, ..., p^2 q - p, q, 2q, 3q, ..., p^2 q - q\}$. The vertex set can be partition into 4-disjoint vertex subsets, $V_1, V_2, V_3, V_4 \in V(\Gamma(Z_{p^2 q}))$, where $V_1 = \{pq, 2pq, 3pq, ..., pq(p-1)\}$, $V_2 = V_2 \setminus V_1 = \{q, 2q, 3q, ..., q(p^2 - 1)\}$, $V_3 = \{p^2, 2p^2, 3p^2, ..., (q-1)p^2\}$, $V_4 = V_4 \setminus V_3 = \{p, 2p, 3p, ..., p(pq-1)\}$. If any two vertices $u, v \in V_1$ are such that $p^2 q$ divides pq and there exists an edge between

u and v then clearly V_1 is an isomorphic to K_{p-1}. Let u and v in V_2 (or V_3 or V_4) are such p^2q does not divides uv, then there exist no edge from the vertex set to itself V_2 (or V_3 or V_4). Clearly, the graph $\Gamma(Z_{p^2q})$ is 4-partite graph. □

Example 2. *Consider the graph of* $\Gamma(Z_{3^2.5}) = \Gamma(Z_{45})$. *The vertex set of* $\Gamma(Z_{45})$ *is* $V = \{3, 5, 6, 9, 10, 12, 15, 18, 20, 21, 24, 25, 27, 30, 33, 35, 36, 39, 42\}$. *The vertex sets* $V_1, V_2, V_3, V_4 \in V(\Gamma(Z_{3^2.5}))$ *can be partition into 4-partite where* $V_1 = \{15, 30\}$, $V_2 = \{5, 10, 20, 25, 35, 40\}$, $V_3 = \{9, 18, 27, 36\}$, $V_4 = \{3, 6, 12, 21, 24, 33, 39, 42\}$. *Theorefore the Figure 2 shows given graph* $\Gamma(Z_{45})$ *is 4-partite graph.*

4. Decomposition of zero divisor graph $\Gamma(Z_{p^2q})$

In this section we investigate the problem of decomposing zero divisor graphs $\Gamma(Z_{p^2q})$ into complete graph K_{p-1} and $\frac{p(p-1)(q-2)}{2}$ copies of C_4, for each p and q distinct prime numbers with $q > p$.

Theorem 4.1. *If* p *is any prime number and* $p > 2$, *then* $\Gamma(Z_{2^2p})$ *is decomposition into 1-copie of* $K_{1,2(p-1)}$ *with* $2(p-1)$ *edges and* $\frac{p-1}{2}$ *copies of* C_4.

Proof. Let p is any prime number and $p > 2$. Let $\Gamma(Z_{2^2p})$ be the non-zero zero divisor graph. The vertex set of $\Gamma(Z_{2^2p})$ is $V = \{2, 4, 6, ..., 2^2p-2, p, 2p, 3p\}$. Let the vertex subsets are $V_1, V_2 \in V$. Let $2p$ be the middle vertex that is adjacent to end vertex and is multiple of 2, then clearly decomposition of edges $E(V) = \{(2p, r)|r = 2, p > 2\}$ or $K_{1,2(p-1)} = \{2p : 2, 4, 6, ..., 2^2p-2\}$. The cardinality of $E(V) = 2(p-1)$. If $E(\Gamma(Z_{2^2p})) \setminus E(K_{1,2(p-1)})$ then there exits complete bipartite graph $K_{2,(p-1)}$ which vertex subsets are $V_1 = \{4, 8, 12, ..., 2^2p-2\}$ and $V_2 = \{p, 3p\}$. That is $|V_1| = p-1$, $|V_2| = 2$, $|V| = |V_1| + |V_2| = 2p-1+2 = 2p+1$. Then decomposition of complete bipartite graph $D(K_{2,(p-1)})$ into cycle of length 4 is as follows.

$$D(K_{2,(p-1)}) = \{(4, p, 8, 3p, 4), (12, p, 16, 3p, 12), (20, p, 24, 3p, 20), ..., (2^2p-8, p, 2^2p-4, 3p, 2^2p-8)\}$$

$$D(K_{2,(p-1)}) = \{C_4, C_4, C_4, ..., C_4(\frac{p-1}{2}\text{times})\}$$

Number of edges is $K_{2,p-1} = 2 \times (p-1)$.
Number of copies of 4 edges is $\frac{2\times(p-1)}{4} = \frac{p-1}{2}$.

Then clearly $K_{2,p-1}$ covers $\frac{p-1}{2}$ copies of C_4. Hence the graph of $\Gamma(Z_{2^2p})$ is decomposition into one copie of $K_{1,2(p-1)}$ and $\frac{p-1}{2}$ copies of C_4. □

Example 3. *Consider* $\Gamma(Z_{2^25}) = \Gamma(Z_{20})$. *The vertex set of* $\Gamma(Z_{20}) = \{2, 4, 5, 6, 8, 10, 12, 14, 15, 16, 18\}$. *The vertex set can be partition into two subsets, namely* V_1 *and* $V_2 \in \Gamma(Z_{20})$ *where* $V_1 = \{5, 10, 15\}$*,* $V_2 = \{2, 4, 6, 8, 12, 14, 16, 18\}$. *Let us take any two vertices* $x, y \in V_1$ *(or* V_2*) which are non-adjacent. Let the decomposition of edges from the graph* $\Gamma(Z_{20})$ *is* $K_{1,8} = \{10 : 2, 4, 6, 8, 12, 14, 16, 18\}$*, then there exits* $K_{2,4}$. $D(K_{2,4}) = \{(4, 5, 8, 15, 4), (12, 5, 16, 15, 12)\}$. *Clearly the Figure 1 shows given graph* $\Gamma(Z_{20})$ *covers 1-star graph* $K_{1,8}$ *and 2 -copies of* C_4.

Theorem 4.2. *If* p *is any prime number and* $p > 3$*, then* $\Gamma(Z_{3^2p})$ *is decomposition into 1-copie of* P_2 *or* K_2 *and* $3(p-1)$ *copies of* C_4.

Proof. Let $\Gamma(Z_{3^2p})$ be the non-zero zero divisor graph. The vertex set of $\Gamma(Z_{3^2p})$ is $V = \{3, 6, 9, ..., 3^2p - 3, p, 2p, 4p, 5p, 7p, 8p\}$. Let vertex subsets are $V_1, V_2, V_3, V_4 \in V$ where $V_1 = \{p, 2p, 4p, 5p, 7p, 8p\}$, $V_2 = \{9, 18, 27, ..., 9(p-1)\} = \{v_1, v_2, v_3, ..., v_{\frac{p-1}{2}}, v_{\frac{p+1}{2}}, ..., v_{p-1}\}$, $V_3 = \{3p, 6p\} = e_1$, $V_4 = V \setminus (V_1 \cup V_2 \cup V_3) = \{3, 6, ..., 3^2p - 3\}$. If the edge e_1 is deleted from the graph of $\Gamma(Z_{3^3p})$ then there exits 3-complete bipartite graph namely $K_{2,p-1}$, $K_{6,p-1}$ and $K_{2,2(p-1)}$. Let as show that $D(K_{n,m})$ be the decomposition of complete bipartite graph into $3(p-1)$ copies of C_4.

$$\begin{aligned}
D(K_{2,p-1}) =&\{(9, 3p, 18, 6p, 9), (27, 3p, 36, 6p, 27), (45, 3p, 54, 6p, 45), ...,\\
&(9p - 18, 3p, 9p - 9, 6p, 9p - 18)\}\\
D(K_{6,p-1}) =&\{(p, v_1, 7p, v_{p-1}, p), (p, v_2, 7p, v_{p-2}, p), (p, v_3, 7p, v_{p-3}, p),\\
&..., (p, v_{\frac{p-1}{2}}, 7p, v_{\frac{p+1}{2}}, p),\\
&(2p, v_1, 6p, v_{p-1}, 2p), (2p, v_2, 6p, v_{p-2}, 2p), (2p, v_3, 6p, v_{p-3}, 2p),\\
&..., (2p, v_{\frac{p-1}{2}}, 6p, v_{\frac{p+1}{2}}, 2p),\\
&(4p, v_1, 5p, v_{p-1}, 4p), (4p, v_2, 5p, v_{p-2}, 4p), (4p, v_3, 5p, v_{p-3}, 4p),\\
&..., (4p, v_{\frac{p-1}{2}}, 5p, v_{\frac{p+1}{2}}, 4p)\}\\
D(K_{2,2(p-1)}) =&\{(3, 3p, 6, 6p, 3), (12, 3p, 15, 6p, 12), (21, 3p, 24, 6p, 21), ...,\\
&(3(3p-1), 3p, 3(3p-2), 6p, 3(3p-1))\}
\end{aligned}$$

Number of decomposition of C_4 is $K_{2,p-1} = \frac{2(p-1)}{4} = \frac{p-1}{2}$.
Number of decomposition of C_4 is $K_{6,p-1} = \frac{6(p-1)}{4} = \frac{3(p-1)}{2}$.

Number of decomposition of C_4 is $K_{2,2(p-1)} = \frac{4(p-1)}{4} = p-1$.

$$\begin{aligned} D(K_{2,p-1}) + D(K_{6,p-1}) + D(K_{2,2(p-1)}) &= \frac{p-1}{2} + \frac{3(p-1)}{2} + p - 1 \\ &= \frac{4(p-1)}{2} + (p-1) \\ &= 3(p-1) \end{aligned}$$

Clearly decomposition of 3-complete bipartite graph is $3(p-1)$ copies of C_4. Hence the graph of $\Gamma(Z_{3^2p})$ is decomposition into 1-copie of P_2 or K_2 and $3(p-1)$ copies of C_4. □

Example of a figure.

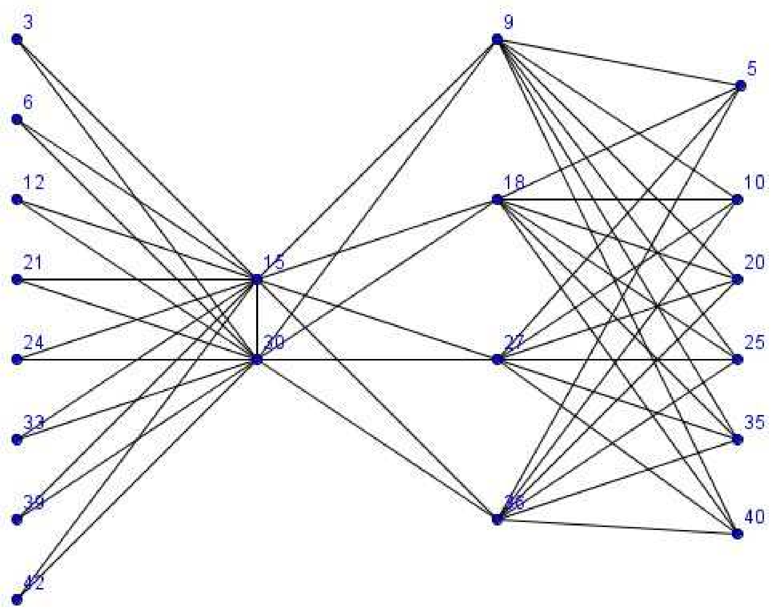

FIGURE 2. $\Gamma(Z_{45})$

Theorem 4.3. *If p is any prime number and $p > 5$, then $\Gamma(Z_{5^2p})$ is decomposition into 1-copie of K_4 with 4 vertices and $10(p-1)$ copies of C_4.*

Proof. Let $\Gamma(Z_{5^2p})$ be the non-zero zero divisor graph. The vertex set of $\Gamma(Z_{5^2p})$ is $V = \{5, 10, 15, ..., 5^2p - 5, p, 2p, 3p, 4p, 6p, 7p, 8p, 9p, 11p, 12p, 13p, 14p, 16p, 17p,$

$18p, 19p, 21p, 22p, 23p, 24p\}$. Let vertex subsets are $V_1, V_2, V_3, V_4 \in V$ where $V_1 = \{p, 2p, 3p, 4p, 6p, 7p, 8p, 9p, 11p, 12p, 13p, 14p, 16p, 17p, 18p, 19p, 21p, 22p, 23p, 24p\}$ $= \{u_1, u_2, u_3, ..., u_{20}\}$, $V_2 = \{25, 50, 75, ..., 25(p-1)\} = \{v_1, v_2, v_3, ..., v_{\frac{p-1}{2}}, v_{\frac{p+1}{2}}, ..., v_{p-1}\}$, $V_3 = \{5p, 10p, 15p, 20p\}$, $V_4 = V \setminus (V_1 \cup V_2 \cup V_3) = \{5, 10, 15, ..., 5^2p-5\} = \{t_1, t_2, t_3, ..., t_{4(p-1)}\}$.

If the edges $\{5p10p, 5p15p, 5p20p, 10p15p, 10p20p, 15p20p\}$ are deleted from the graph of $\Gamma(Z_{5^3p})$ then there exits 3-complete bipartite graph namely $K_{4,p-1}$, $K_{20,p-1}$ and $K_{4,4(p-1)}$. Let $D(K_{n,m})$ denotes decomposition of complete bipartite graph into $10(p-1)$ copies of C_4.

$$\begin{aligned}
D(K_{4,p-1}) =& \{(5p, v_1, 20p, v_{p-1}, 5p), (5p, v_2, 20p, v_{p-2}, 5p), (5p, v_3, 20p, v_{p-3}, 5p),\\
& ..., (5p, v_{\frac{p-1}{2}}, 20p, v_{\frac{p+1}{2}}, 5p),\\
& (10p, v_1, 15p, v_{p-1}, 10p), (10p, v_2, 15p, v_{p-2}, 10p), (10p, v_3, 15p, v_{p-3},\\
& 10p), ..., (10p, v_{\frac{p-1}{2}}, 15p, v_{\frac{p+1}{2}}, 10p)\}\\
D(K_{20,p-1}) =& \{(u_1, v_1, u_{20}, v_{p-1}, u_1), (u_1, v_2, u_{20}, v_{p-2}, u_1), (u_1, v_3, u_{20}, v_{p-3}, u_1),\\
& ..., (u_1, v_{\frac{p-1}{2}}, u_{20}, v_{\frac{p+1}{2}}, u_1)\\
& (u_2, v_1, u_{19}, v_{p-1}, u_2), (u_2, v_2, u_{19}, v_{p-2}, u_2), (u_2, v_3, u_{19}, v_{p-3}, u_2),\\
& ..., (u_2, v_{\frac{p-1}{2}}, u_{19}, v_{\frac{p+1}{2}}, u_2)\\
& (u_3, v_1, u_{18}, v_{p-1}, u_3), (u_3, v_2, u_{18}, v_{p-2}, u_3), (u_3, v_3, u_{18}, v_{p-3}, u_3),\\
& ..., (u_3, v_{\frac{p-1}{2}}, u_{18}, v_{\frac{p+1}{2}}, u_3)\\
& (u_4, v_1, u_{17}, v_{p-1}, u_4), (u_4, v_2, u_{17}, v_{p-2}, u_4), (u_4, v_3, u_{17}, v_{p-3}, u_4),\\
& ..., (u_4, v_{\frac{p-1}{2}}, u_{17}, v_{\frac{p+1}{2}}, u_4)\\
& (u_5, v_1, u_{16}, v_{p-1}, u_5), (u_5, v_2, u_{16}, v_{p-2}, u_5), (u_5, v_3, u_{16}, v_{p-3}, u_5),\\
& ..., (u_5, v_{\frac{p-1}{2}}, u_{16}, v_{\frac{p+1}{2}}, u_5)\\
& (u_6, v_1, u_{15}, v_{p-1}, u_6), (u_6, v_2, u_{15}, v_{p-2}, u_6), (u_6, v_3, u_{15}, v_{p-3}, u_6),\\
& ..., (u_6, v_{\frac{p-1}{2}}, u_{15}, v_{\frac{p+1}{2}}, u_6)\\
& (u_7, v_1, u_{14}, v_{p-1}, u_7), (u_7, v_2, u_{14}, v_{p-2}, u_7), (u_7, v_3, u_{14}, v_{p-3}, u_7),\\
& ..., (u_7, v_{\frac{p-1}{2}}, u_{14}, v_{\frac{p+1}{2}}, u_7)
\end{aligned}$$

$$
\begin{aligned}
&(u_8, v_1, u_{13}, v_{p-1}, u_8), (u_8, v_2, u_{13}, v_{p-2}, u_8), (u_8, v_3, u_{13}, v_{p-3}, u_8),\\
&..., (u_8, v_{\frac{p-1}{2}}, u_{13}, v_{\frac{p+1}{2}}, u_8)\\
&(u_9, v_1, u_{12}, v_{p-1}, u_9), (u_9, v_2, u_{12}, v_{p-2}, u_9), (u_9, v_3, u_{12}, v_{p-3}, u_9),\\
&..., (u_9, v_{\frac{p-1}{2}}, u_{12}, v_{\frac{p+1}{2}}, u_9)\\
&(u_{10}, v_1, u_{11}, v_{p-1}, u_{10}), (u_{10}, v_2, u_{11}, v_{p-2}, u_{10}), (u_{10}, v_3, u_{11}, v_{p-3}, u_{10}),\\
&..., (u_{10}, v_{\frac{p-1}{2}}, u_{11}, v_{\frac{p+1}{2}}, u_{10})\}
\end{aligned}
$$

$$
\begin{aligned}
D(K_{4,4(p-1)}) =&\{(5p, t_1, 20p, t_{4(p-1)}, 5p), (5p, t_2, 20p, t_{4(p-1)-1}, 5p),\\
&(5p, t_3, 20p, t_{4(p-1)-2}, 5p), ..., (5p, t_{\frac{4(p-1)}{2}}, 20p, t_{\frac{4(p-1)}{2}+1}, 5p),\\
&(10p, t_1, 15p, t_{4(p-1)}, 10p), (10p, t_2, 15p, t_{4(p-1)-1}, 10p),\\
&(10p, t_3, 15p, t_{4(p-1)-2}, 10p), ..., (10p, t_{\frac{4(p-1)}{2}}, 15p, t_{\frac{4(p-1)}{2}+1}, 10p)\}
\end{aligned}
$$

Number of decomposition of C_4 in $K_{4,p-1} = \frac{4(p-1)}{4} = p - 1$.
Number of decomposition of C_4 in $K_{20,p-1} = \frac{20(p-1)}{4} = 5(p - 1)$.
Number of decomposition of C_4 in $K_{4,4(p-1)} = \frac{16(p-1)}{4} = 4(p - 1)$.

$$
\begin{aligned}
D(K_{2,p-1}) + D(K_{6,p-1})+&D(K_{2,2(p-1)})\\
&=p - 1 + 5(p - 1) + 4(p - 1) = 10(p - 1)
\end{aligned}
$$

Clearly decomposition of 3-complete bipartite graph is $10(p - 1)$ copies of C_4. Hence the graph of $\Gamma(Z_{5^2p})$ is decomposition into 1- copie of complete graph K_4 with 4 vertices and $10(p - 1)$ copies of C_4. □

Example of a figure.

Theorem 4.4. *If p is any prime number and $p > 7$, then $\Gamma(Z_{7^2p})$ is decomposition of 1-copie of K_6 with 6 vertices and $21(p - 1)$ copies of C_4.*

Proof. Let $\Gamma(Z_{7^2p})$ be the non-zero zero divisor graph. The vertex set of $\Gamma(Z_{7^2p})$ is
$V = \{7, 14, 21, ..., 7^2p - 7, p, 2p, 3p, 4p, 5p, 6p, 8p, 9p, 10p, 11p, 12p, 13p,$
$15p, 16p, 17p, 18p, 19p, 20p, 22p, 23p, 24p, 25p, 25p, 26p, 27p, 29p, 30p, 31p, 32p, 33p,$
$34p, 36p, 37p, 38p, 39p, 40p, 41p, 43p, 44p, 45p, 46p, 47p, 48p\}$.

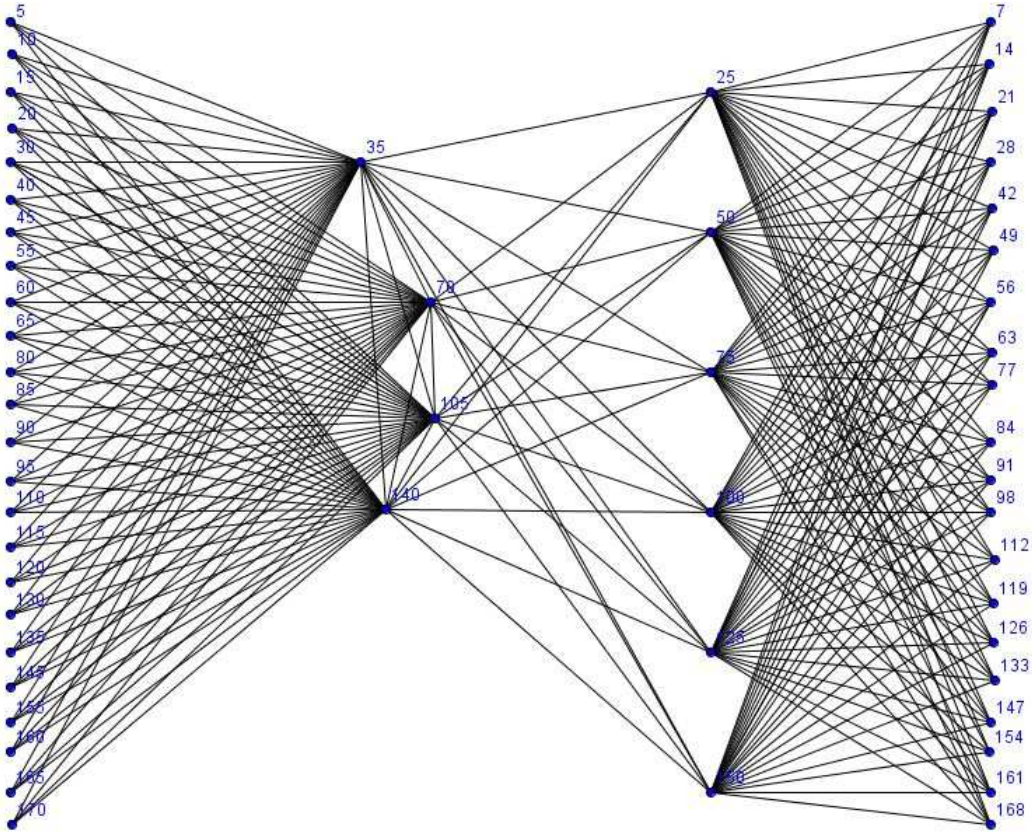

FIGURE 3. $\Gamma(Z_{175})$

Let vertex subsets are $V_1, V_2, V_3, V_4 \in V$ where

$$\begin{aligned}
V_1 =&\{p, 2p, 3p, 4p, 5p, 6p, 8p, 9p, 10p, 11p, 12p, 13p, 15p, 16p, 17p, 18p, 19p, 20p,\\
&22p, 23p, 24p, 25p, 25p, 26p, 27p, 29p, 30p, 31p, 32p, 33p, 34p, 36p, 37p, 38p,\\
&39p, 40p, 41p, 43p, 44p, 45p, 46p, 47p, 48p\}\\
=&\{u_1, u_2, u_3, ..., u_{42}\},\\
V_2 =&\{49, 98, 147, ..., 49(p-1)\}\\
=&\{v_1, v_2, v_3, ..., v_{\frac{p-1}{2}}, v_{\frac{p+1}{2}}, ..., v_{p-1}\},\\
V_3 =&\{7p, 14p, 21p, 28p, 35p, 42p\},\\
V_4 =&V \setminus (V_1 \cup V_2 \cup V_3)\\
=&\{7, 14, 21, ..., 7^2p-7\}\\
=&\{t_1, t_2, t_3, ..., t_{3(p-1)}, t_{3(p-1)+1}, ..., t_{6(p-1)}\}.
\end{aligned}$$

If the edges $\{7p14p, 7p21p, 7p28p, 7p35p, 7p42p, 14p21p, 14p28p, 14p35p, 14p42p, 21p28p, 21p35p, 21p42p, 28p35p, 28p42p, 35p42p\}$ are deleted from the graph of $\Gamma(Z_{7^3p})$ then there exits the 3-complete bipartite graph namely $K_{6,p-1}$, $K_{42,p-1}$ and $K_{6,6(p-1)}$.

$$
\begin{aligned}
D(K_{6,p-1}) =& \{(7p, v_1, 42p, v_{p-1}, 7p), (7p, v_2, 42p, v_{p-2}, 7p), (7p, v_3, 42p, v_{p-3}, 7p),\\
& ..., (7p, v_{\frac{p-1}{2}}, 42p, v_{\frac{p+1}{2}}, 7p)\\
& (14p, v_1, 35p, v_{p-1}, 14p), (14p, v_2, 35p, v_{p-2}, 14p), (14p, v_3, 35p, v_{p-3}, 14p),\\
& ..., (14p, v_{\frac{p-1}{2}}, 35p, v_{\frac{p+1}{2}}, 14p)\\
& (21p, v_1, 28p, v_{p-1}, 21p), (21p, v_2, 28p, v_{p-2}, 21p), (21p, v_3, 28p, v_{p-3}, 21p),\\
& ..., (21p, v_{\frac{p-1}{2}}, 28p, v_{\frac{p+1}{2}}, 21p)\}
\end{aligned}
$$

$$
\begin{aligned}
D(K_{42,p-1}) =& \{(u_1, v_1, u_{42}, v_{p-1}, u_1), (u_1, v_2, u_{42}, v_{p-2}, u_1), (u_1, v_3, u_{42}, v_{p-3}, u_1),\\
& ...., (u_1, v_{\frac{p-1}{2}}, u_{42}, v_{\frac{p+1}{2}}, u_1),\\
& (u_2, v_1, u_{41}, v_{p-1}, u_2), (u_2, v_2, u_{41}, v_{p-2}, u_2), (u_2, v_3, u_{41}, v_{p-3}, u_2),\\
& ..., (u_2, v_{\frac{p-1}{2}}, u_{41}, v_{\frac{p+1}{2}}, u_2)\\
& (u_3, v_1, u_{40}, v_{p-1}, u_3), (u_3, v_2, u_{40}, v_{p-2}, u_3), (u_3, v_3, u_{40}, v_{p-3}, u_3),\\
& ..., (u_3, v_{\frac{p-1}{2}}, u_{40}, v_{\frac{p+1}{2}}, u_3)\\
& \vdots\\
& (u_{21}, v_1, u_{22}, v_{p-1}, u_{21}), (u_{21}, v_2, u_{22}, v_{p-2}, u_{21}), (u_{21}, v_3, u_{22}, v_{p-3}, u_{21}),\\
& ..., (u_{21}, v_{\frac{p-1}{2}}, u_{22}, v_{\frac{p+1}{2}}, u_{21})\}
\end{aligned}
$$

$$
\begin{aligned}
D(K_{6,6(p-1)}) = \Big\{ & (7p, t_1, 42p, t_{6(p-1)}, 7p), (7p, t_2, 42p, t_{6(p-1)-1}, 7p),\\
& (7p, t_3, 42p, t_{6(p-1)-2}, 7p), ..., (7p, t_{3(p-1)}, 42p, t_{3(p-1)+1}, 7p)\\
& (14p, t_1, 35p, t_{6(p-1)}, 14p), (14p, t_2, 35p, t_{6(p-1)-1}, 14p),\\
& (14p, t_3, 35p, t_{6(p-1)-2}, 14p), ..., (14p, t_{3(p-1)}, 35p, t_{3(p-1)+1}, 14p)\\
& (21p, t_1, 28p, t_{6(p-1)}, 21p), (21p, t_2, 28p, t_{6(p-1)-1}, 21p),\\
& (21p, t_3, 28p, t_{6(p-1)-2}, 21p), ..., (21p, t_{3(p-1)}, 28p, t_{3(p-1)+1}, 21p)\Big\}
\end{aligned}
$$

Number of decomposition of C_4 in $K_{6,p-1} = \frac{6(p-1)}{4} = \frac{3(p-1)}{2}$.
Number of decomposition of C_4 in $K_{42,p-1} = \frac{42(p-1)}{4} = \frac{21(p-1)}{2}$.

Number of decomposition of C_4 in $K_{6,6(p-1)} = \frac{36(p-1)}{4} = 9(p-1)$.

$$\begin{aligned} D(K_{6,p-1}) + D(K_{42,p-1}) + D(K_{6,6(p-1)}) \\ &= \frac{3(p-1)}{2} + \frac{21(p-1)}{2} + 9(p-1) \\ &= 12(p-1) + 9(p-1) \\ &= 21(p-1). \end{aligned}$$

Clearly decomposition of 3 complete bipartite graph is $21(p-1)$ copies of C_4. Hence the graph of $\Gamma(Z_{7^2p})$ is decomposition into 1-complete graph K_6 with 6 vertices and $21(p-1)$ copies of C_4. □

Theorem 4.5. *If p and q are distint prime numbers with $p < q$, then $\Gamma(Z_{p^2q})$ is decomposition of 1-copie of K_{p-1} with $p-1$ vertices and $\frac{p(p-1)(q-1)}{2}$ copies of C_4.*

Proof. Let $\Gamma(Z_{5^2p})$ be the non-zero zero divisor graph. The vertex set of $\Gamma(Z_{5^2p})$ is $V = \{p, 2p, 3p, ..., p^2q - p, q, 2q, 3q, ..., (p^2 - 1)q\}$. Using the above theorem $\Gamma(Z_{p^2q})$ is 1- complete graph and cycle of length 4. Therefore decomposition of 3 complete bipartite graph is $\frac{p(p-1)(q-1)}{2}$ copies of C_4. Hence the graph of $\Gamma(Z_{p^2q})$ is decomposition into 1-complete graph K_{p-1} with $p-1$ vertices and $\frac{p(p-1)(q-1)}{2}$ copies of C_4. □

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DEPARTMENT OF MATHEMATICS
VELLORE INSTITUTE OF TECHNOLOGY,VELLORE
VELLORE, TAMIL NADU - 632014, INDIA.

DEPARTMENT OF MATHEMATICS
VELLORE INSTITUTE OF TECHNOLOGY,VELLORE
VELLORE, TAMIL NADU - 632014, INDIA.
Email address: kuppan.a@vit.ac.in, ravisankar.j@vit.ac.in