

EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR DISCRETE THREE POINT BOUNDARY VALUE PROBLEM WITH FRACTIONAL ORDER

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ABSTRACT. In this paper, we deal with the existence and uniqueness of solutions for discrete fractional order three point boundary value problem (BVP) of the form

$$\begin{aligned}\Delta^v w(s) &= -f(s+v-1, w(s+v-1)), \\ w(v-3) &= \psi(w), \Delta w(v-3) = 0, w(v+b+1) = \phi(w),\end{aligned}$$

where $s \in [0, b+1]_{N_0}$, $f : [v-3, v-2, v-1, \dots, v+b+1]_{N_{v-3}} \times R \rightarrow [0, +\infty]$ is a continuous function, $\psi, \phi : C([v-3, v+b+1]_{N_{v-3}}) \rightarrow R$ are given functions and Δ^v is a discrete fractional operator with $2 < v \leq 3$. We prove the existence and uniqueness of solutions by contraction mapping principle and the Brouwer theorem and also we conclude with examples to illustrate the results.

1. INTRODUCTION

Fractional calculus is a branch of mathematics, which is as old as calculus. It deals with differentiation and integration of arbitrary orders. Its origin can be traced back to the end of the 17-th century, the time when Newton and Leibniz developed the foundations of differential and integral calculus. The subject started developing since then with the pioneering contributions from Leibniz,

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Euler, Bernoulli, Abel, Laplace, Lagrange, Fourier and many others. By the middle of nineteenth century, satisfactory expressions for generalizations of integration/differentiation to arbitrary orders were given through the researchers of Liouville, Riemann, Grunwald and Letnikov. During last few decades very interesting and novel applications of these concepts have been found in the areas of control theory, viscoelasticity, aerodynamics, diffusion, thermodynamics, electrical circuits, engineering, electro chemistry, blood flow problems, polymer rheology, enzyme kinetics, electrical circuits and biology etc., see in [1–3, 5]. Christopher S. Goodrich in [4, 5] considered a discrete fractional order two point BVP of the existence, uniqueness and positive solutions by contraction mapping principle, the Brouwer theorem and the Krasnosel'skii theorem. Atici et al. in [6, 7] investigated the Leibniz rule, summation by parts, initial value problem and transform method in fractional calculus. In this article, we consider the discrete fractional order three point BVP of the form

$$(1.1) \quad \begin{aligned} \Delta^v w(s) &= -f(s+v-1, w(s+v-1)), \\ w(v-3) &= \psi(w), \Delta w(v-3) = 0, w(v+b+1) = \phi(w), \end{aligned}$$

where $s \in [0, b+1]_{N_0}$, $f : [v-3, v-2, v-1, \dots, v+b+1]_{N_{v-3}} \times R \rightarrow [0, +\infty]$ is a continuous function, $\psi, \phi : C([v-3, v+b+1]_{N_{v-3}}) \rightarrow R$ are given functions and Δ^v is a discrete fractional operator with $2 < v \leq 3$. We prove the existence and uniqueness of solutions by the Brouwer theorem and contraction mapping principle.

The plan of this paper is as follows. Some basic definitions and lemmas are provided in section 2. In section 3, The contraction mapping principle and Brouwer theorem are presented. In section 4, we present the examples to illustrating as the application of our main results.

2. PRELIMINARIES

In this section, some basic definitions and lemma's are provided.

Definition 2.1. Let $v > 0$. The v -th fractional sum is defined by

$$\Delta^{-v} f(s) = \Delta^{-v} f(s, a) := \frac{1}{\Gamma(v)} \sum_{\xi=a}^{s-v} (s-\xi-1)^{(v-1)} f(\xi)$$

for $s \in \{a + \nu, a + \nu + 1, \dots\} = N_{a+\nu}$. Let $\nu > 0$. The ν -th fractional difference is defined by $\Delta^\nu f(s) = \Delta^N \Delta^{\nu-N} f(s)$ where $s \in N_{a+\nu}$ and

$$s^{(\nu)} := \frac{\Gamma(s+1)}{\Gamma(s+1-\nu)}.$$

Lemma 2.1. [3, 5], Let s and ν be any numbers $s^{(\nu)}$ and $s^{(\nu-1)}$,

$$\Delta s^{(\nu)} = \nu s^{(\nu-1)}.$$

Lemma 2.2. [3, 5], $s \leq r$, then $s^{(\mu)} \leq r^{(\mu)}$ for any $\mu > 0$.

Lemma 2.3. [5], Let $0 \leq N-1 < \nu \leq N$. Then

$$\Delta^{-\nu} \Delta^\nu w(s) = w(s) + c_1 s^{(\nu-1)} + c_2 s^{(\nu-2)} + \dots + c_N s^{(\nu-N)}$$

for some $c_i \in R$ with $1 \leq i \leq N$.

Lemma 2.4. [5] For s and ξ , for with both $(s - \xi - 1)^{(\nu)}$ and $(s - \xi - 2)^{(\nu)}$

$$\Delta_\xi [(s - \xi - 1)^{(\nu)}] = -\nu(s - \xi - 2)^{(\nu-1)}.$$

Lemma 2.5. The function

$\frac{s^{(\nu-2)}}{\Gamma(\nu-1)} + \frac{s^{(\nu-3)}}{\Gamma(\nu-2)} - \frac{s^{(\nu-1)}}{(b+3)\Gamma(\nu-1)} - \frac{s^{(\nu-1)}}{(b+3)(b+4)\Gamma(\nu-2)}$ is strictly decreasing in s , for $s \in [v-3, v+k+1]_{N_{v-3}}$.

$$\min_{s \in [v-3, v+k+1]_{N_{v-3}}} \left[\frac{s^{(\nu-2)}}{\Gamma(\nu-1)} + \frac{s^{(\nu-3)}}{\Gamma(\nu-2)} - \frac{s^{(\nu-1)}}{(b+3)\Gamma(\nu-1)} - \frac{s^{(\nu-1)}}{(b+3)(b+4)\Gamma(\nu-2)} \right] = 0$$

and

$$\max_{s \in [v-3, v+k+1]_{N_{v-3}}} \left[\frac{s^{(\nu-2)}}{\Gamma(\nu-1)} + \frac{s^{(\nu-3)}}{\Gamma(\nu-2)} - \frac{s^{(\nu-1)}}{(b+3)\Gamma(\nu-1)} - \frac{s^{(\nu-1)}}{(b+3)(b+4)\Gamma(\nu-2)} \right] = 1.$$

Lemma 2.6. Let $2 < \nu < 3$, $h : [v-2, v+b+1]_{N_{v-2}} \rightarrow R$ and $\psi, \phi : R^{b+4} \rightarrow R$. Function w is a solution for BVP of FODE as in (1.1) which is given by

$$(2.1) \quad \Delta^\nu w(s) = -h(s + \nu - 1),$$

$$(2.2) \quad w(v-3) = \psi(w), \Delta w(v-3) = 0, w(v+b+1) = \phi(w),$$

where $w(s)$ if and only if $s \in [0, b+1]_{N_0}$, for $s \in [v-3, v+b+1]_{N_{v-3}}$ has

$$\begin{aligned}
 w(s) = & -\frac{1}{\Gamma(v)} \sum_{\xi=0}^{t-v} (t-\xi-1)^{(v-1)} h(\xi+v-1) \\
 & + \psi(w) \left[\frac{s^{(v-2)}}{\Gamma(v-1)} + \frac{s^{(v-3)}}{\Gamma(v-2)} - \frac{s^{(v-1)}}{(b+3)\Gamma(v-1)} - \frac{s^{(v-1)}}{(b+3)(b+4)\Gamma(v-2)} \right] \\
 & + \phi(w) \left[\frac{s^{(v-1)}}{(v+b+1)^{(v-1)}} \right] \\
 (2.3) \quad & + \frac{s^{(v-1)}}{\Gamma(v)(v+b+1)^{(v-1)}} \sum_{\xi=0}^{b+1} (v+b-\xi)^{(v-1)} h(\xi+v-1).
 \end{aligned}$$

3. EXISTENCE AND UNIQUENESS OF SOLUTIONS

In this section, We prove that fractional order three point BVP (1.1) has atleast one solution. Let w is a solution of (1.1) if and only if w is the operator $S : R^{b+4} \rightarrow R$, has

$$\begin{aligned}
 (Sw)(s) = & -\frac{1}{\Gamma(v)} \sum_{\xi=0}^{s-v} (s-\xi-1)^{(v-1)} f(s+v-1, w(s+v-1)) \\
 & + \psi(w) \left[\frac{s^{(v-2)}}{\Gamma(v-1)} + \frac{s^{(v-3)}}{\Gamma(v-2)} - \frac{s^{(v-1)}}{(b+3)\Gamma(v-1)} - \right. \\
 & \left. \frac{s^{(v-1)}}{(b+3)(b+4)\Gamma(v-2)} \right] + \phi(w) \left[\frac{s^{(v-1)}}{(v+b+1)^{(v-1)}} \right] \\
 (3.1) \quad & + \frac{s^{(v-1)}}{\Gamma(v)(v+b+1)^{(v-1)}} \sum_{\xi=0}^{b+1} (v+b-\xi)^{(v-1)} f(s+v-1, w(s+v-1)).
 \end{aligned}$$

for $s \in [v-3, v+b+1]_{N_{v-3}}$.

Theorem 3.1. Assume that $f(s, w)$, $\psi(w)$ and $\phi(w)$ are Lipschitz in w and we define $\|w\| = \max_{s \in [v-3, v+b+1]_{N_{v-2}}} |w(s)|$. Then $\omega, \delta, \gamma > 0$,

$$\begin{aligned}
 |f(s, w_1) - f(s, w_2)| & \leq \omega \|w_1 - w_2\|, \\
 |\psi(w_1) - \psi(w_2)| & \leq \delta \|w_1 - w_2\|, \\
 |\phi(w_1) - \phi(w_2)| & \leq \gamma \|w_1 - w_2\|,
 \end{aligned}$$

for w_1, w_2 any function and $[v-3, v+b+1]_{N_{v-3}}$ are defined,

$$(3.2) \quad 2\omega \frac{\Gamma(v+b+2)}{\Gamma(v+1)\Gamma(b+2)} + (\delta + \gamma) < 1$$

holds, then the fractional order three point BVP (1.1) has a unique solution.

Proof. We prove that S is a contraction mapping and w_1 and w_2 are given functions

$$\begin{aligned} \|Sw_1 - Sw_2\| &\leq \omega \|w_1 - w_2\| \max_{s \in [v-3, v+b+1]_{N_{v-3}}} \left[\frac{1}{\Gamma(v)} \sum_{\xi=0}^{s-v} (s-\xi-1)^{(v-1)} \right] \\ &+ \delta \|w_1 - w_2\| \max_{s \in [v-3, v+b+1]_{N_{v-3}}} \left[\frac{s^{(v-2)}}{\Gamma(v-1)} + \frac{s^{(v-3)}}{\Gamma(v-2)} - \right. \\ &\quad \left. \frac{s^{(v-1)}}{(b+3)\Gamma(v-1)} - \frac{s^{(v-1)}}{(b+3)(b+4)\Gamma(v-2)} \right] \\ &+ \gamma \|w_1 - w_2\| \max_{s \in [v-3, v+b+1]_{N_{v-3}}} \left[\frac{s^{(v-1)}}{(v+b+1)^{(v-1)}} \right] \\ (3.3) \quad &+ \omega \|w_1 - w_2\| \max_{s \in [v-3, v+b+1]_{N_{v-2}}} \left[\frac{s^{(v-1)}}{\Gamma(v)(v+b+1)^{(v-1)}} \sum_{\xi=0}^{b+1} (v+b-\xi)^{(v-1)} \right] \end{aligned}$$

By application of Lemma (2.4), we simplify the inequality (3.3), we obtain

$$\begin{aligned} \frac{1}{\Gamma(v)} \sum_{\xi=0}^{s-v} (s-\xi-1)^{(v-1)} &= \frac{1}{\Gamma(v)} \left[-\frac{1}{v} (s-\xi)^{(v)} \right]_{\xi=0}^{s-v+1} \\ &= \frac{1}{\Gamma(v+1)} [s^{(v)}] \\ &= \frac{1}{\Gamma(v+1)} \frac{\Gamma(s+1)}{\Gamma(s+1-v)} \\ (3.4) \quad \frac{1}{\Gamma(v)} \sum_{\xi=0}^{s-v} (s-\xi-1)^{(v-1)} &= \frac{\Gamma(v+b+2)}{\Gamma(v+1)\Gamma(b+2)} \end{aligned}$$

From the Lemma (2.2), we simplify the inequality (3.3), we obtain

$$(3.5) \quad \frac{s^{(v-1)}}{(v+b+1)^{(v-1)}} \leq \frac{(v+b+1)^{(v-1)}}{(v+b+1)^{(v-1)}} = 1.$$

Similar, we simplify the inequality (3.3), we obtain

$$\begin{aligned}
 \frac{s^{(v-1)}}{\Gamma(v)(v+b+1)^{(v-1)}} \sum_{\xi=0}^{b+1} (v+b-\xi)^{(v-1)} &\leq \frac{1}{\Gamma(v)} \sum_{\xi=0}^{b+1} (v+b-\xi)^{(v-1)} \\
 &= \frac{1}{\Gamma(v)} \left[-\frac{1}{v} (v+b-\xi)^{(v)} \right]_{\xi=0}^{b+1} \\
 (3.6) \quad \frac{s^{(v-1)}}{\Gamma(v)(v+b+1)^{(v-1)}} \sum_{\xi=0}^{b+1} (v+b-\xi)^{(v-1)} &= \frac{\Gamma(v+b+2)}{\Gamma(v+1)\Gamma(b+2)}.
 \end{aligned}$$

From the Lemma (2.5) and simplify the inequality (3.3), we have

$$\begin{aligned}
 \left[\frac{s^{(v-2)}}{\Gamma(v-1)} + \frac{s^{(v-3)}}{\Gamma(v-2)} - \frac{s^{(v-1)}}{(b+3)\Gamma(v-1)} - \frac{s^{(v-1)}}{(b+3)(b+4)\Gamma(v-2)} \right] &= \frac{\Gamma(v-2)}{\Gamma(v-2)} \\
 (3.7) \quad \left[\frac{s^{(v-2)}}{\Gamma(v-1)} + \frac{s^{(v-3)}}{\Gamma(v-2)} - \frac{s^{(v-1)}}{(b+3)\Gamma(v-1)} - \frac{s^{(v-1)}}{(b+3)(b+4)\Gamma(v-2)} \right] &= 1.
 \end{aligned}$$

we obtain (3.4),(3.5), (3.6) and (3.7) in (3.3), we obtain

$$\begin{aligned}
 \|Sw_1 - Sw_2\| &\leq \omega \|w_1 - w_2\| \left[\frac{\Gamma(v+b+2)}{\Gamma(v+1)\Gamma(b+2)} \right] + \delta \|w_1 - w_2\| (1) + \\
 &\quad + \gamma \|w_1 - w_2\| (1) \\
 &\quad + \omega \|w_1 - w_2\| \left[\frac{\Gamma(v+b+2)}{\Gamma(v+1)\Gamma(b+2)} \right] \\
 &\leq 2\omega \|w_1 - w_2\| \left[\frac{\Gamma(v+b+2)}{\Gamma(v+1)\Gamma(b+2)} \right] + \delta \|w_1 - w_2\| + \gamma \|w_1 - w_2\| \\
 (3.8) \quad \|Sw_1 - Sw_2\| &\leq \left[2\omega \frac{\Gamma(v+b+2)}{\Gamma(v+1)\Gamma(b+2)} + (\delta + \gamma) \right] \|w_1 - w_2\|.
 \end{aligned}$$

Then condition (3.2) holds. We find that the three point BVP (1.1) has a unique solution. $\square$

We apply the Brouwer fixed point theorem.

Theorem 3.2. Suppose $M_1 > 0$ is a constant and such that $f(s, w)$, $\psi(w)$ and $\phi(w)$ satisfies the inequality

$$(3.9) \quad \max_{(s,w) \in [v-3, v+b+1]_{N_{v-3}} \times [-M_1, M_1]} |f(s, w)| \\ \leq \frac{M_1}{\left[2 \left(\frac{\Gamma(v+b+2) + \Gamma(v+1)\Gamma(b+2)}{\Gamma(v+1)\Gamma(b+2)} \right) \right]}$$

$$(3.10) \quad \max_{w \in [-M_1, M_1]} |\psi(w)| \\ \leq \frac{M_1}{\left[2 \left(\frac{\Gamma(v+b+2) + \Gamma(v+1)\Gamma(b+2)}{\Gamma(v+1)\Gamma(b+2)} \right) \right]},$$

$$(3.11) \quad \max_{w \in [-M_1, M_1]} |\phi(w)| \\ \leq \frac{M_1}{\left[2 \left(\frac{\Gamma(v+b+2) + \Gamma(v+1)\Gamma(b+2)}{\Gamma(v+1)\Gamma(b+2)} \right) \right]}.$$

Then the three point BVP (1.1) has atleast one solution w_0 satisfying $|w_0(s)| \leq M_1$, $s \in [v-3, v+b+1]_{N_{v-3}}$.

Proof. We assume the Banach Space $B := \{w \in R^{b+4} : \|w\| \leq M_1\}$. S is defined as (3.1) and S is a continuous operator. We show that $S : B \rightarrow B$, whenever $\|Sw\| \leq A$.

Assume the inequalities (3.9), (3.10) and (3.11) holds.

$$(3.12) \quad \Phi := \frac{M_1}{\left[2 \left(\frac{\Gamma(v+b+2) + \Gamma(v+1)\Gamma(b+2)}{\Gamma(v+1)\Gamma(b+2)} \right) \right]},$$

$$\begin{aligned}
\|Sw\| \leq & \max_{s \in [v-3, v+b+1]_{N_{v-3}}} \frac{1}{\Gamma(v)} \sum_{\xi=0}^{s-v} (s-\xi-1)^{(v-1)} |f(\xi+v-1, w(\xi+v-1))| + \\
& \max_{s \in [v-3, v+b+1]_{N_{v-3}}} |\psi(w)| \\
& \left[\frac{s^{(v-2)}}{\Gamma(v-1)} + \frac{s^{(v-3)}}{\Gamma(v-2)} - \frac{s^{(v-1)}}{(b+3)\Gamma(v-1)} - \frac{s^{(v-1)}}{(b+3)(b+4)\Gamma(v-2)} \right] \\
& + \max_{s \in [v-3, v+b+1]_{N_{v-3}}} |\phi(w)| \left[\frac{s^{(v-1)}}{(v+b+1)^{(v-1)}} \right] \\
& + \max_{s \in [v-3, v+b+1]_{N_{v-3}}} \left[\frac{s^{(v-1)}}{\Gamma(v)(v+b+1)^{(v-1)}} \sum_{\xi=0}^{b+1} (v+b-\xi)^{(v-1)} \right] \\
& \quad \times |f(\xi+v-1, w(\xi+v-1))|.
\end{aligned}$$

$$\begin{aligned}
\|Sw\| \leq & \Phi \max_{s \in [v-3, v+b+1]_{N_{v-3}}} \frac{1}{\Gamma(v)} \sum_{\xi=0}^{s-v} (s-\xi-1)^{(v-1)} \\
& + \Phi \max_{s \in [v-3, v+b+1]_{N_{v-3}}} \left[\frac{s^{(v-2)}}{\Gamma(v-1)} + \right. \\
& \left. \frac{s^{(v-3)}}{\Gamma(v-2)} - \frac{s^{(v-1)}}{(b+3)\Gamma(v-1)} - \frac{s^{(v-1)}}{(b+3)(b+4)\Gamma(v-2)} \right] \\
& + \Phi \max_{s \in [v-3, v+b+1]_{N_{v-3}}} \left[\frac{s^{(v-1)}}{(v+b+1)^{(v-1)}} \right] \\
(3.13) \quad & + \Phi \max_{s \in [v-3, v+b+1]_{N_{v-3}}} \left[\frac{s^{(v-1)}}{\Gamma(v)(v+b+1)^{(v-1)}} \sum_{\xi=0}^{b+1} (v+b-\xi)^{(v-1)} \right].
\end{aligned}$$

By application of Lemma (2.4), and simplify the inequality (3.13), we obtain

$$\begin{aligned}
& \frac{1}{\Gamma(v)} \sum_{\xi=0}^{s-v} (s-\xi-1)^{(v-1)} + \frac{s^{(v-1)}}{\Gamma(v)(v+b+1)^{(v-1)}} \sum_{\xi=0}^{b+1} (v+b-\xi)^{(v-1)} \\
& = \frac{\Gamma(v+b+2)}{\Gamma(v+1)\Gamma(b+2)} + \frac{\Gamma(v+b+2)}{\Gamma(v+1)\Gamma(b+2)} \\
(3.14) \quad & = \frac{2\Gamma(v+b+2)}{\Gamma(v+1)\Gamma(b+2)}.
\end{aligned}$$

By application of Lemma (2.4), and simplify the inequality (3.13), we obtain

$$(3.15) \quad \frac{s^{(v-2)}}{\Gamma(v-1)} + \frac{s^{(v-3)}}{\Gamma(v-2)} - \frac{s^{(v-1)}}{(b+3)\Gamma(v-1)} - \frac{s^{(v-1)}}{(b+3)(b+4)\Gamma(v-2)} \leq \frac{\Gamma(v-1)}{\Gamma(v-1)} = 1$$

From the Lemma (2.2), we simplify the expression on the third term of the right hand side as (3.13) of the inequality

$$(3.16) \quad \frac{s^{(v-1)}}{(v+b+1)^{(v-1)}} \leq \frac{(v+b+1)^{(v-1)}}{(v+b+1)^{(v-1)}} = 1.$$

By substituting (3.14), (3.15) and (3.16) into (3.13), we obtain

$$(3.17) \quad \begin{aligned} \|Sw\| &\leq \Phi \left[\frac{2\Gamma(v+b+2)}{\Gamma(v+1)\Gamma(b+2)} \right] + \Phi + \Phi \\ &\leq \Phi \left[\frac{2\Gamma(v+b+2)}{\Gamma(v+1)\Gamma(b+2)} \right] + 2\Phi \\ \|Tw\| &\leq 2\Phi \left[\frac{\Gamma(v+b+2) + \Gamma(v+1)\Gamma(b+2)}{\Gamma(v+1)\Gamma(b+2)} \right]. \end{aligned}$$

By inserting (3.12) in (3.17), we obtain

$$(3.18) \quad \|Sw\| \leq 2 \frac{A}{2 \left[\frac{\Gamma(v+b+2) + \Gamma(v+1)\Gamma(b+2)}{\Gamma(v+1)\Gamma(b+2)} \right]} \cdot \left[\frac{\Gamma(v+b+2) + \Gamma(v+1)\Gamma(b+2)}{\Gamma(v+1)\Gamma(b+2)} \right] = M_1.$$

We deduce that $T : B \rightarrow B$, $Sw_0 = w_0$, $w_0 \in B$ then Brouwer fixed point theorem that there exists. The function w_0 satisfies $|w_0(s)| \leq M_1$ for each $t \in [v-3, v+b+1]_{N_{v-3}}$. Hence the map S is a fixed point. $\square$

4. NUMERICAL EXAMPLES

In this article, we present the two examples to illustrating as the application of our results.

Example 1. Suppose that $v = \frac{5}{2}, b = 3$. Let $f(s, w(s)) = \frac{|\cos w(s)|}{s^2 + 95}$, $\psi(w) = \frac{1}{85} \|\sin w\|$ and $\phi(w) = \frac{1}{75} \|\sin w\|$. Then (1.1) becomes

$$-\Delta^{\frac{5}{2}} w(s) = \frac{|\cos w(s)|}{s^2 + 95}, w\left(-\frac{1}{2}\right) = \frac{1}{85} \|\sin w\|,$$

$$\Delta w\left(-\frac{1}{2}\right) = 0, w\left(\frac{13}{2}\right) = \frac{1}{75} \|\sin w\|.$$

Case (i). Suppose that $\omega = \frac{1}{95}, \delta = \frac{1}{85}$ and $\gamma = \frac{1}{75}$. The inequality (3.2) is

$$\begin{aligned} Q &= 2\omega \frac{\Gamma(v+b+2)}{\Gamma(v+1)\Gamma(b+2)} + (\delta + \gamma) \leq \\ &\leq 2\left(\frac{1}{95}\right) \frac{\Gamma\left(\frac{15}{2}\right)}{\Gamma\left(\frac{7}{2}\right)\Gamma(5)} + \left(\frac{1}{85} + \frac{1}{75}\right) \leq 0.5190 < 1. \end{aligned}$$

Case (ii). Suppose that $\omega = \frac{1}{85}, \delta = \frac{1}{75}$ and $\gamma = \frac{1}{65}$. The inequality (3.2) is

$$\begin{aligned} Q &= 2\omega \frac{\Gamma(v+b+2)}{\Gamma(v+1)\Gamma(b+2)} + (\delta + \gamma) \leq \\ &\leq 2\left(\frac{1}{85}\right) \frac{\Gamma\left(\frac{15}{2}\right)}{\Gamma\left(\frac{7}{2}\right)\Gamma(5)} + \left(\frac{1}{75} + \frac{1}{65}\right) \leq 0.5807 < 1. \end{aligned}$$

Case (iii). Suppose that $\omega = \frac{1}{75}, \delta = \frac{1}{65}$ and $\gamma = \frac{1}{55}$. The inequality (3.2) is

$$\begin{aligned} Q &= 2\omega \frac{\Gamma(v+b+2)}{\Gamma(v+1)\Gamma(b+2)} + (\delta + \gamma) \leq \\ &\leq 2\left(\frac{1}{75}\right) \frac{\Gamma\left(\frac{15}{2}\right)}{\Gamma\left(\frac{7}{2}\right)\Gamma(5)} + \left(\frac{1}{65} + \frac{1}{55}\right) \leq 0.6592 < 1. \end{aligned}$$

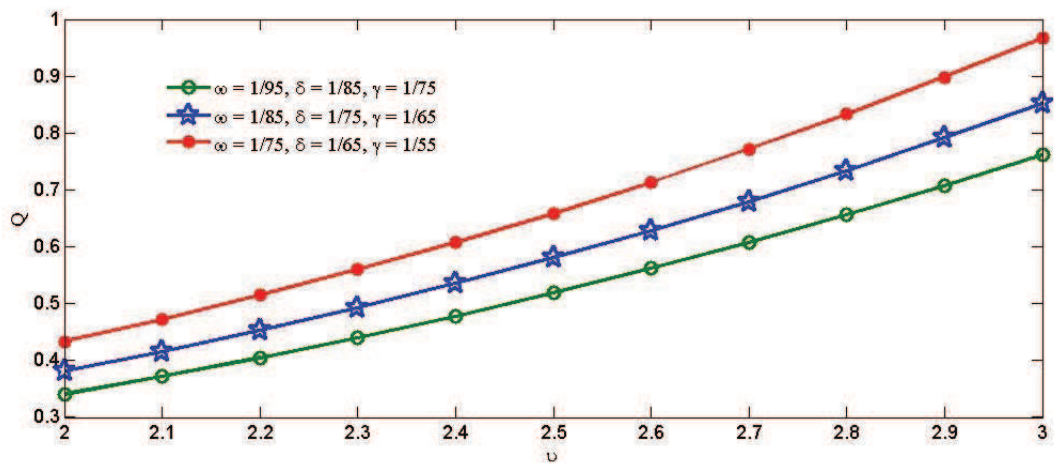
Example 2. Suppose that $v = \frac{5}{2}, b = 3$ and $M_1 = 100$. Let $f(s, w(s)) = \frac{s}{39} e^{\frac{w^2}{5}}$, $\psi(w) = \frac{1}{4} \|\cos w\|$ and $\phi(w) = \frac{1}{2} \|\cos w\|$. Then (1.1) becomes

$$-\Delta^{\frac{5}{2}} w(s) = \frac{s}{39} e^{\frac{w^2}{5}}, w\left(-\frac{1}{2}\right) = \frac{1}{4} \|\cos w\|,$$

$$\Delta w\left(-\frac{1}{2}\right) = 0, w\left(\frac{13}{2}\right) = \frac{1}{2} \|\cos w\|.$$

TABLE 1. Various values of v of example 1.

v	$Q < 1$		
	$\omega = \frac{1}{95}, \delta = \frac{1}{85}, \gamma = \frac{1}{75}$	$\omega = \frac{1}{85}, \delta = \frac{1}{75}, \gamma = \frac{1}{65}$	$\omega = \frac{1}{75}, \delta = \frac{1}{65}, \gamma = \frac{1}{55}$
2.0	0.3409	0.3817	0.4336
2.1	0.3719	0.4164	0.4729
2.2	0.4052	0.4535	0.5150
2.3	0.4407	0.4932	0.5600
2.4	0.4786	0.5356	0.6080
2.5	0.5190	0.5807	0.6592
2.6	0.5620	0.6288	0.7136
2.7	0.6077	0.6798	0.7715
2.8	0.6561	0.7340	0.8329
2.9	0.7075	0.7914	0.8980
3.0	0.7619	0.8522	0.9669

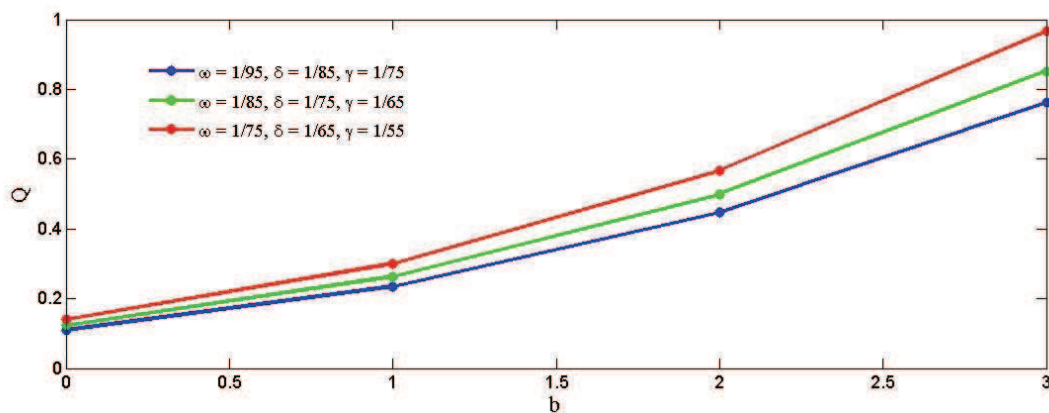
FIGURE 1. Various values of v with $Q < 1$.

The Banach space $B := \{w \in R^7 : \|w\| \leq 100\}$.

$$\begin{aligned}
 & \frac{M_1}{\left[2^{\left(\frac{\Gamma(v+b+2)+\Gamma(v+1)\Gamma(b+2)}{\Gamma(v+1)\Gamma(b+2)}\right)}\right]} = \\
 & = \frac{100}{\left[2^{\left(\frac{\Gamma\left(\frac{15}{2}\right)+\Gamma\left(\frac{7}{2}\right)\Gamma(5)}{\Gamma\left(\frac{7}{2}\right)\Gamma(5)}\right)}\right]} > \frac{100}{48.9219} \approx 2.0441
 \end{aligned}$$

TABLE 2. Various values of b of example 1.

	$Q < 1$		
b	$\omega = \frac{1}{95}, \delta = \frac{1}{85}, \gamma = \frac{1}{75}$	$\omega = \frac{1}{85}, \delta = \frac{1}{75}, \gamma = \frac{1}{65}$	$\omega = \frac{1}{75}, \delta = \frac{1}{65}, \gamma = \frac{1}{55}$
0	0.1093	0.1228	0.1402
1	0.2356	0.2640	0.3002
2	0.4462	0.4993	0.5669
3	0.7619	0.8522	0.9669

FIGURE 2. Various values of b with $Q < 1$.

It is clear that $|f(s, w)| \leq 0.1666 < 2.0441$, $|\psi(w)| \leq 0.25 < 2.0441$, $|\phi(w)| \leq 0.5 < 2.0441$. Therefore f, ψ and ϕ satisfy the condition.

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