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FIXED POINT THEOREMS IN PARAMETRIC METRIC SPACE FOR CONTRACTION MAPPINGS

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ABSTRACT. Parametric Metric Space (P.M.S) is a well known generalization of Metric Space (MS) which was introduced and studied by Hussian (a new approach to M.S.) in 2014. In the present paper, we extended fixed point theorems (F.P.T) of Zamfirescu and Hardy-Rogers based on injective mapping to Parametric Metric Space (P.M.S) using contraction conditions. Moreover, we presented an examples to validate our result.

1. GENERAL INTRODUCTION AND PRELIMINARIES

Real analysis is the most important branch of mathematics. Among several branches of real analysis, functional analysis is the most important part of real analysis. Functional analysis is divided into two parts: linear and non-linear. F.P.T is an important part of non-linear functional analysis since 1960. F.P.T has various applications in field of pure and applied mathematics as well as in physical, economic, differential equation, integral equation and life sciences.It has emerged as one of the major links between abstracts mathematics and its applications. It is used in differential equations, integral theory, artificial intelligence, computer science, decision making, medical diagnosis, neural network, social science and many other related areas. F.P.T deals with the

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classical approach to the exact solution and to check the stability of the system. F.P.T has fascinated lots of researchers since 1922, with the celebration of Banach (Polish mathematician) contraction principle [23] which provided a constructive method to find a fixed point of a mapping. Banach contraction principal states: A mapping $T : X \to X$ defined on M.S (X, δ) is called contraction if $\delta(Ta, Tb) \leq k\delta(a, b)\ \forall\ a, b \in X\quad 0 < k < 1$. Since then this theorem became an important tool for the development of nonlinear analysis. Contraction conditions begins by studying Banach contraction principle which are utilized in various F.P.T for some generalized M.S. Various F.P.T were attained by expanding Banach contraction principal [2, 3, 16, 19, 21–23].

Kannan [15] proved that a mapping $T : X \to X$ defined on M.S (X, δ) is called contraction mapping if $\delta(Ta, Tb) \leq k[\delta(a, Ta) + \delta(b, Tb)]\,\forall\, a, b \in X\quad 0 < k < 1/2$ for operators that need not be continuous. Further, Chatterjea [3], proved a F.P.T for discontinuous mapping, which is actually a kind of dual of Kannan mapping. Some of the generalizations of M.S are cone metric spaces, partial metric spaces, P.M.S, "parametric b-metric space"(P.B.M.S) etc. [1, 6, 10, 11, 13, 14, 17, 20, 21] introducing and modifying the metric axioms.

Hussain et al. [12] studied and introduced the notion of P.M.S, and later on gave generalized P.M.S and introduced P.B.M.S which is combination of M.S, as well as, b-M.S [13]. In the present paper we proved fixed point theorems on P.M.S for expansive mapping. We generalized the fixed point theorems of Zamfirescu [32] and Hardy-Rogers [12] to their parametric versions in complete metric space.

Definition 1.1. *Let X be a non empty set and $T_p : X \times X \times (0, \infty) \to (0, \infty)$ be a map on X such that $\forall\ a, b, c \in X$ and $t > 0$*

(a) $T_p(a, b, t) = 0$ *if and only if* $a = b$

(b) $T_p(a, b, t) = T_p(b, a, t)$

(c) $T_p(a, b, t) \leq T_p(a, c, t) + T_p(c, b, t)$

Then pair (X, δ) is called P.M.S.

Definition 1.2. *Let* X *be a non empty set and (X, δ) be a* P.M. S *and let $\{a_i\}$ be a sequence in X.*

(a) *If* $\log_{i\to\infty}(a_i, a, t) = 0 \Rightarrow \log_{i\to\infty}[a_i = a]$, *for all $t > 0$ then sequence $\{a_i\}_{i=1}^{\infty}$ converses $a \in X$.*

(*b*) *If* $\log_{n\to\infty}(a_i, a_j, t) = 0$ *for all* $t > 0$ *then sequence* $\{a_i\}_{i=1}^{\infty}$ *is called Cauchy sequence.*

(*c*) *If every Cauchy sequence is convergent, then* P.M *S* (X, δ) *is a complete P.M.S.*

(*d*) *Let* (X, δ) *be a* $P.MS$ *and* $T_p : X \to X$ *be a mapping, then we say T is a continuous mapping in X, if for any sequence*$\{a_i\}_{i=1}^{\infty} \in X$ *such that* $\log_{i\to\infty} a_i = x \quad \Rightarrow \log_{i\to\infty} T_p a_i = T_p a$.

2. Main result

Theorem 2.1. *Let* (X, δ) *be a complete* $\mathrm{P_M MS}$ *and* $T_{\mathrm{p}} : X \to X$ *be an injective mapping satisfying the condition*

$$\delta(T_p a, T_p b, t) \leq \alpha_1 \delta(a, b, t) + \alpha_2 \delta(a, T_p a, t) + \alpha_3 \delta(b, T_p b, t) + \alpha_4 \delta(a, T_p b, t) + \alpha_5 \delta(b, T_p a, t) \tag{2.1.1}$$

$\forall t \in [0, 1); \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 > 0; a, b \in X \& a \neq b$ *have a fixed point if* $\alpha_1 + \alpha_2 + \alpha_2 + 2\alpha_4 < 1$ *and moreover a unique point if* $\alpha_1 + \alpha_4 + \alpha_{\mathrm{s}} < 1$.

Proof. Let $a_0 \in X_2$. Define iterative sequence $[a_i]_{i=1}^{\infty}$ as follows: $T_p a_i = a_{i+1}$ for $i = 1, 2, 3, \ldots$. If for some i, $T_p a_i = a_i$, then a_i is the fixed point. Otherwise $T_p a_i \neq a_i$, using inequality (2.1.1),

$$\begin{aligned}
\delta(a_{i+1}, a_{i+2}, t) &= \delta(T_p a_i, T_p a_{i+1}, t) \\
&\leq \alpha_1 \delta(a_i, a_{i+1}, t) + \alpha_2 \delta(a_i, T_p a_i, t) + \alpha_3 \delta(a_{i+1}, T_p a_{i+1}, t) \\
&\quad + \alpha_4 \delta(a_i, T_p a_{i+1}, t) + \alpha_5 \delta(a_{i+1}, T_p a_i, t) \\
&\leq \alpha_1 \delta(a_i, a_{i+1}, t) + \alpha_2 \delta(a_i, a_{i+1}, t) + \alpha_3 \delta(a_{i+1}, a_{i+2}, t) \\
&\quad + \alpha_4 \delta(a_i, a_{i+2}, t) + \alpha_5 \delta(a_{i+1}, a_{i+1}, t) \\
&\leq \alpha_1 \delta(a_i, a_{i+1}, t) + \alpha_2 \delta(a_i, a_{i+1}, t) + \alpha_2 \delta(a_{i+1}, a_{i+2}, t) \\
&\quad + \alpha_4 [\delta(a_i, a_{i+1}, t) + \delta(a_{i+1}, a_{i+2}, t)]
\end{aligned}$$

$$(1 - \alpha_3 - \alpha_4) \delta(a_{i+1}, a_{i+2}, t) \leq (\alpha_1 + \alpha_2 + \alpha_4) \delta(a_i, a_{i+1}, t)$$

$$\delta(a_{i+1}, a_{i+2}, t) \leq \tfrac{(\alpha_1 + \alpha_2 + \alpha_4)}{(1 - \alpha_3 - \alpha_4)} \delta(a_i, a_{i+1}, t)$$

$$\delta(a_{i+1}, a_{i+2}, t) \leq k \delta(a_i, a_{i+1}, t)$$

Here $k = \frac{(a_1 + a_2 + a_4)}{(1 - a_3 - a_4)} < 1 \Rightarrow \alpha_1 + \alpha_2 + \alpha_3 + 2a_4 < 1$.

Continuing iterations up to i times $\delta(a_{i+1}, a_{i+2}, t) \leq k^i \delta(a_0, a_1, t)$.

As we know if $\{a_i\}_{i\to\infty}$ be a sequence in $\mathrm{P.MS}(X, \delta)$ such that $\delta(a_{i+1}, a_{i+2}, t) \leq k^i \delta(a_0, a, t)$, $\forall t \in [0, 1)$, and $t = 1, 2, 3, \ldots$. Then $\{a_i\}_{i+\infty}$ is a Cauchy sequence in (X, δ). Since (X, δ) is a complete P.M.S $\{a_i\}_{i\to\infty}$ converses.

Let $a^+ \in X$, then $lim_{n\to\infty}x_i \to x^*$. Again T_p is continuous, therefore

$$T_pa^* = T_p\left(lim_{n\to\infty}a_n\right) = lim_{n\to\infty}T_pa_i = a^* \Rightarrow T_pa^* = a^*$$

which implies that T_p has a fixed point $T_pa^* = a^*$ in X.

Now we will show that a^4 is unique. For that suppose b^* is another fixed point, i.e., $T_pb^* = b^*$. Therefore by inequality $(2.1.1)$ we have

$$\begin{aligned}\delta\left(T_pa^*, T_pb^*, t\right) \leq \alpha_1\delta\left(a^*, b^*, t\right) + \alpha_2\delta\left(a^*T_pa^*, t\right) + \alpha_3\delta\left(b^*, T_pb^*, t\right)\\ +\alpha_4\delta\left(a^*, T_pb^*, t\right) + \alpha_5\delta\left(b^*, T_pa^*, t\right),\end{aligned}$$

$$\begin{aligned}\delta\left(a^*, b^*, t\right) &\leq \alpha_1\delta\left(a^*, b^*, t\right) + \alpha_2\delta\left(a^*, b^*, t\right) + \alpha_2\delta\left(b^*, b^*, t\right)\\ &\quad +\alpha_4\delta\left(a^*, b^*, t\right) + \alpha_5\delta\left(b^*, a^*, t\right)\\ \delta\left(a^*, b^*, t\right) &\leq \alpha_1\delta\left(a^*, b^*, t\right) + \alpha_4\delta\left(a^*, b^*, t\right) + \alpha_5\delta\left(a^*, b^*, t\right)\\ \delta\left(a^*, b^*, t\right) &\leq \left(\alpha_1 + \alpha_4 + \alpha_5\right)\delta\left(a^*, b^*, t\right)\end{aligned}$$

$$\left(1 - \alpha_1 - \alpha_4 - \alpha_5\right)\delta\left(a^*, b^*, t\right) \leq 0,$$

implying $a^* = b^*$. Since $\alpha_1 + \alpha_4 + \alpha_5 < 1$, we have that a^* and b^* are not different point but are same.

Hence a^4 is unique. □

Theorem 2.2. *Let (X, T_{p}) be a complete P.MS and $T_p : X \to X$ be an injective mapping satisfying condition*

$$(T_pa, T_pb, t) \leq \alpha \max\left[\delta(a, b, t), \frac{\delta\left(a, \tau_pa, t\right) + \delta\left(b, \tau_pb, t\right)}{2}, \frac{\delta\left(a, \tau_pb, t\right) + \delta\left(b, \tau_pa, t\right)}{2}\right]$$

for all $t \in [0, 1); \alpha, \beta, \gamma > 0; x, y \in X$ and $a \neq b$. Then T_p has a fixed point if $\alpha + \beta + \gamma < 1$ and moreover a unique fixed point if $\alpha + \gamma < 1$.

Proof. Let $a_0 \in X$. Define iterative sequence $\{a_i\}_{i=1}^{\infty}$ as follows: $T_pa_i = a_{i+1}$ for $i = 1, 2, 3, \ldots$. If for some i, $T_pa_i = a_i$, then a_i is the fixed point. Otherwise $T_pa_i \neq a_i$, and then

$$\begin{aligned}\delta\left(a_{i+1}, a_{i+2}, t\right) &= \delta\left(T_pa_i, T_pa_{i+1}, t\right)\\ &\leq \alpha \max\left\{\delta\left(a_i, a_{i+1}, t\right), \tfrac{\delta(a_i, T_pa_i, t) + \delta(a_{i+1}, T_pa_{i+1}, t)}{2}, \tfrac{\delta(a_i, T_pa_{i+1}, t) + \delta(a_{i+1}, T_pa_i, t)}{2}\right\}\\ &\leq \alpha \max\left\{\delta\left(a_i, a_{i+1}, t\right), \tfrac{\delta(a_i, a_{i+1}, t) + \delta(a_{i+1}, a_{i+2}, t)}{2}, \tfrac{\delta(a_i, a_{i+2}, t) + \delta(a_{i+1}, a_{i+1}, t)}{2}\right\}\\ &\leq \alpha \max\left\{\delta\left(a_i, a_{i+1}, t\right), \tfrac{\delta(a_i, a_{i+1}, t) + \delta(a_{i+1}, a_{i+2}, t)}{2}, \tfrac{\delta(a_i, a_{i+1}, t) + \delta(a_{i+1}, a_{i+2}, t)\}}{2}\right\}\end{aligned}$$

which implies $\delta\left(a_{i+1}, a_{i+2}, t\right) \leq \alpha \max\left(a_i, a_{i+1}, t\right)$.

Therefore, by successive iteration $\delta\left(a_{i+1}, a_{i+2}, t\right) \leq \alpha^i\delta\left(a_0, a_1, t\right)$.

As we know, if $\{a_i\}_{i\to\infty}$ is a sequence in $\mathrm{P.MS}(X,\delta)$ such that $\delta\left(a_{i+1},a_{i+2},t\right)\leq\alpha^i\delta\left(a_0,a_1,t\right)$ for all $t\in[0,1)$ and $i=1,2,3,\ldots$, then $\{a_i\}_{i\to\infty}$ is a Cauchy sequence in (X,δ).

Since (X,δ) is a complete $\mathrm{P_1M.\ S}$ $\{a_i\}_{i\to\infty}$ converges.

Let $a^*\in X$, then $a\to a^*$. Again T_p is continuous, therefore

$$T_pa^*=T_p\left(lim_{n\to\infty}a_i\right)=\lim_{n\to\infty}T_pa_i=a^*\Rightarrow T_pa^*=a^*.$$

and further, T_p has a fixed point $T_pa^*=a^*$ in X.

Now, we will show that a^* is unique. For that suppose y^* is another fixed point therefore $T_py^*=y^*$. Therefore by the condition of the theorem we have

$$\delta\left(T_pa^*,T_pb^*,t\right)$$
$$\leq\alpha\max\left\{\delta\left(a^*,b^*,t\right),\frac{\delta(a^*,T_pa^*,t)+\delta(b^*,T_pb^*,t)}{2},\frac{\delta(a^*,T_pb^*,t)+\delta(b^*,T_pa^*,t)}{2}\right\}$$
$$\delta\left(T_pa^*,T_pb^*,t\right)\leq\alpha\max\left\{\delta\left(a^*,b^*,t\right),\frac{\delta(a^*,a^*,t)+\delta(b^*,b^*,t)}{2},\frac{\delta(a^*,b^*,t)+\delta(b^*,a^*,t)}{2}\right\}$$
$$\delta\left(T_pa^*,T_pb^*,t\right)\leq\alpha\max\left\{\delta\left(a^*,b^*,t\right)\right\}$$
$$\delta\left(a^*,b^*,t\right)\leq\alpha\delta\left(a^*,b^*,t\right),$$

implying $(1-\alpha)\delta\left(a^*,b^*,t\right)\leq0$ and $\Rightarrow(1-\alpha)\delta\left(a^*,b^*,t\right)\leq0$, i.e., $\delta\left(a^*,b^*,t\right)=0$ since $\alpha>1\Rightarrow a^*=b^*$.

Hence T_p has a unique point. □

Example 1. *Let (X,δ) be a complete P.M.S, where $T_p:R^+\to R^+$ is a mapping defined as $\delta(a,b,t)=t|a-b|^q$ such that $a_n=1+\frac{1}{n}$ and $b_n=1+\frac{2}{n}$. Therefore*

$$\delta\left(a_n,b_n,t\right)=t\left|a_n-b_n\right|^q=t\left|\left(1+\tfrac{1}{n}\right)-\left(1+\tfrac{2}{n}\right)\right|^q$$
$$=t\left|\tfrac{1}{n}-\tfrac{2}{n}\right|^q=t\left|-\tfrac{1}{n}\right|^q=t\left(\tfrac{1}{n}\right)^q=t\tfrac{1}{n^q},$$
$$\log_{n\to\infty}\delta\left(a_n,b_n,t\right)=\log_{n\to\infty}t\tfrac{1}{n^q}=t\log_{n\to\infty}\tfrac{1}{n9}=0\ \textit{for}\ t>0$$

implying $\log_{n\to\infty}\delta\left(a_n,b_n,t\right)\to0$ as both $a_n=1+\frac{1}{n}$ and $b_n=1+\frac{2}{n}$ tends to 1 as $n\to\infty$.

Hence 1 is the fixed point. It satisfies all the conditions of complete parametric metric space for $t>0$ and Theorems 2.1 and 2.2.

3. Conclusion

The aim of this paper is to verify that existing theorems in complete metric space are true or not for parametric metric space if a parameter $t>0$ is added under the contraction conditions.

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