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AXISYMMETRIC PROBLEM: IN FRACTIONAL ORDER GENERALIZED THERMOELASTIC MEDIUM

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ABSTRACT. We study the axisymmetric problem: Generalized thermoelastic medium in fractional order for Green-Lindsay [2] theory. Here various components of normal displacement v_z , normal force stress t_{zz} and temperature distribution T are obtained in converted domain by applying Laplace (LT) and Hankel transforms (HT). The result obtained can be applied to some particular problem subjected to normal source and radial source.

1. INTRODUCTION

Nomenclature.

- τ_0, ν_0 : Thermal relaxation time.
- α : Conductivity.
- T: Temperature.
- p: Hydrostatic initial stress.(HIS)
- e: $\operatorname{div} \vec{u}$.
- F: Young's modulus.
- σ : Poission ratio.

In classical theory of thermo-elasticity infinite speed of thermal waves was used. Practically it is not acceptable. It was observed that in various cases finite

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speed of thermal waves are used. To overcome this problem Lord and Shulmam [1] presented theory of GT with one relaxation time.

Later on Lindsay and Green [2] presented the theory of two relaxation time without effecting the Fourier's Law of heat conduction. Barber [3] contemplated thermo-elastic removals and worries due to sources of heat lying outside the half plane. Sherief [4]obtained components of stress and temperature disseminations in a thermo-elastic medium due to continuous source. H.sherief, El-Sayed [5] developed the new theory of elasticity by using the method of fractional calculus. A Ezzat [6] also worked on in theory of heat conduction with fractional order. Sharma et al. [7-8] and [10-13] explored the disturbance due to mechanical and thermal sources in GT half-space.

There are various reasons of HIS in the medium it may be created due to difference in temperature, shot sticking and due to other external sources, change in gravity, and so on. Ailawalia and Kumar [9] has also studied the roational effect under HIS and gravity.

The current paper is related with the problem of Axi-symmetric: Generalized thermo elastic medium in fractional order with (HIS) and conductivity. Graphically we can demonstrated the Impact of hydrostatic initial stress and conductivity.

2. BASIC EQUATIONS

The governing equations are:

(2.1)

$$\sigma_{fj} = \rho \ddot{i} \ddot{u}_i,$$

$$\sigma_{ij} = -p \left(\delta_{ij} + w_{ij}\right) + 2\mu e_{ij} + \lambda e \delta_{ij} - v \left(1 + v_0 \frac{\partial}{\partial t}\right) T \delta_{ij}$$

$$P^{\alpha - 1} K^* \left(n^* + t_1 \frac{\partial}{\partial t}\right) T_{,ii} = \rho C_p \left(n_1 \frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right) T$$

$$+ \nu T_0 \left(n_1 \frac{\partial}{\partial t} + \tau_0 n_0 \frac{\partial^2}{\partial t^2}\right) e,$$

where P^{α} is Riemann-Liouville fractional integral,

$$P^{\alpha}g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\eta)^{\alpha-1} g(\eta) d\eta,$$

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for $0 < \alpha < 2$ and $P^{\alpha}g(t) = g(t)$ for $\alpha = 0$. Here $\Gamma(\alpha)$ is the gamma function and $0 \le \alpha < 1$ for weak conductivity and $\alpha = 1$, for normal conductivity and $1 < \alpha \le 2$, for strong conductivity.

FORMULATION OF THE PROBLEM

We consider an isotropic, homogeneous, generalized thermoelastic medium with HIS at tempertaure T_Q , with cylindrical polar coordinates (r, θ, z) by considering origin at z = 0. z- axis is normal to medium. The plane is considered to be axis-symmetric. We assume displacement vector $\vec{v} = (v_r, 0, v_z)$, quantities remain independent of θ . The above consideration would reduce to the following equation,

$$(2.3) \qquad \begin{aligned} &(\lambda+2\mu)\frac{\partial^2 v_r}{\partial r^2} + \left(\lambda+\mu-\frac{p}{2}\right)\frac{\partial^2 v_z}{\partial r\partial z} + \left(\mu+\frac{p}{2}\right)\frac{\partial^2 v_r}{\partial z^2} \\ &+ \frac{1}{r}(\lambda+2\mu)\frac{\partial v_r}{\partial r} - (\lambda+2\mu)\frac{v_r}{r^2} - v\left(1+\vartheta_0\frac{\partial}{\partial t}\right)\frac{\partial T}{\partial r} = \rho\frac{\partial^2 v_r}{\partial t^2} \\ &(\lambda+2\mu)\frac{\partial^2 v_z}{\partial z^2} + \left(\lambda+\mu+\frac{p}{2}\right)\frac{\partial^2 v_r}{\partial r\partial z} + \left(\mu-\frac{p}{2}\right)\frac{\partial^2 v_z}{\partial r^2} \\ &+ \frac{1}{r}\left(\lambda+\mu+\frac{p}{2}\right)\frac{\partial v_r}{\partial z} + \frac{1}{r}\left(\mu-\frac{p}{2}\right)\frac{\partial v_z}{\partial r} - v\left(1+\vartheta_0\frac{\partial}{\partial t}\right)\frac{\partial T}{\partial z} = \rho\frac{\partial^2 u_z}{\partial t^2} \end{aligned}$$

$$(2.4) \qquad I^{\alpha-1}K^*\left(n^*+t_1\frac{\partial}{\partial t}\right)\nabla^2 T = \rho C_E\left(n_1\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)T + \\ &VT_0\left(n_1\frac{\partial}{\partial t}+\tau_0n_0\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}\right). \end{aligned}$$

Introducing dimensionless variables defined by

(2.5)
$$r' = \frac{\omega^*}{c_0}r, \quad z' = \frac{\omega^*}{c_0}z, \quad v'_i = \frac{\rho\alpha_0\omega^*}{vT_0}v_i, \quad t = \omega^*$$

(2.6)
$$\begin{aligned} \tau_0' &= \omega^* \tau_0, \quad v_0' &= \omega^* v_0, \quad T' = \frac{T}{T_0}, \quad \sigma_{ij}' = \frac{\sigma_{ij}}{\sqrt{T_0}} \\ p' &= \frac{p}{vT_0}, \quad \omega^* = \frac{\rho C_p c_0^2}{K^*}, \quad (\lambda + 2\mu) = \rho c_0^2. \end{aligned}$$

In equations (2.4-2.6), in order to get the dimensionless equation we define displacement component v_r, v_z in terms of ϕ, ψ ,

(2.7)
$$v_r = \frac{\partial \phi}{\partial r} + \frac{\partial^2 \psi}{\partial r \partial z}, \quad v_z = \frac{\partial \phi}{\partial z} - \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r}.$$

In the resulting dimensionless equations we get

(2.8)

$$\nabla^2 \phi - \frac{\phi}{r^2} - \left(1 + v_0 \frac{\partial}{\partial t}\right) T = \frac{\partial^2 \phi}{\partial t^2}$$
$$\nabla^2 \psi - \frac{\psi}{r^2} = a_1 \frac{\partial^2 \psi}{\partial t^2}$$
$$P^{\alpha - 1} \left(n^* + t_1 \frac{\partial}{\partial t}\right) \nabla^2 T = \left(n_1 \frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right) T + \varepsilon \left(n_1 \frac{\partial}{\partial t} + \tau_0 n_0 \frac{\partial^2}{\partial t^2}\right) \nabla^2 \phi.$$

Apply LT with respect to ' t ' and HT with respect to " r " it will convert tinto s domain and r into q domain. In equation (2.9)-(2.11) we get

(2.9)
$$\left(\frac{d^2}{dz^2} - q^2 - s^2\right)\tilde{\phi} - (1 + v_0 s)\tilde{T} = 0$$
$$\left(\frac{d^2}{dz^2} - q^2 - a_1 s^2\right)\tilde{\psi} = 0$$
$$\left(\frac{d^2}{dz^2} - q^2 - a_2\right)\tilde{T} - a_3\left(\frac{d^2}{dz^2} - q^2\right)\tilde{\phi} = 0.$$

Eliminating \tilde{T} from equation (2.10) and (2.12), we get the resulting equation

(2.10)
$$\left\{ D^4 + MD^2 + N \right\} \tilde{\phi} = 0$$

where

$$(2.11) M = a_6 + a_4 - a_3 a_5, N = a_4 a_6 - a_3 a_5 q^2$$

with

(2.12)
$$a_{1} = \frac{\rho x_{0}^{2}}{\mu + \frac{v T_{0} p}{2}}, \quad a_{2} = \frac{n_{1} s^{\alpha} + \tau_{0} s^{\alpha+1}}{n^{*} + t_{1} s}$$
$$a_{3} = \frac{\varepsilon (n_{1} s^{\alpha} + n_{0} \tau_{0} s^{\alpha+1})}{n^{*} + t_{1} s}, \quad a_{4} = -(q^{2} + s^{2})$$
$$a_{5} = (1 + v_{0} s), \quad a_{6} = -(q^{2} + a_{2})$$

and

(2.13)
$$\varepsilon = \frac{v^2 T_0}{\rho \omega^* K^*}.$$

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The equation is having solution which satisfies the condition of radiation $\tilde{\phi}, \tilde{\psi}, \tilde{T} \to 0$ as $z \to \infty$ are

$$\begin{split} \tilde{\psi} &= A_1 e^{-\beta_1^2} \\ \tilde{\phi} &= A_2 e^{-P_2 z} + A_3 e^{-p_3 z} \\ \tilde{T} &= a_2^* A_2 e^{-P_2 z} + a_3^* A_3 e^{-p_3 z}, \end{split}$$

where $p_1^2 = a_1 s^2 + q^2$ and p_2^2, p_3^2 are roots of (2.13) and b_i^* is coupling constant having value

$$b_i^* = \frac{p_i^2 + a_4}{a_5}, \quad i = 2, 3.$$

3. BOUNDARY CONDITIONS

Case I: Loading along normal direction.

When a load F(r,t) put at the interface z = 0 at half space in normal direction are

$$\sigma_{x=} = -F_1(r,t), \quad \sigma_{rz} = 0, \quad T = 0.$$

Here

$$\begin{split} \Delta &= -b_2^* \left(r_1 s_3 - r_3 s_1 \right) + b_3^* \left(r_1 s_2 - r_2 s_1 \right), \quad \Delta_1 = q_1 \left(b_3^* s_2 - b_2^* s_3 \right) \\ \Delta_2 &= -b_3^* s_1 q_1, \quad \Delta_3 = b_2^* s_1 q_1, \quad q_1 = p - F_1 \left(r_7 s \right) \\ s_1 &= -\left[\frac{\left(\mu + \frac{v T_{0p}}{2} q p_1^2 + q^3 \frac{\mu - \frac{v T_{0p}}{2}}{\rho c_0^2} \right], \quad r_1 = \frac{2\mu q^2 p_1}{\rho c_0^2} \\ s_i &= \frac{2\mu q p_i}{\rho c_0^2}, \quad r_i = p_i^2 - \frac{\lambda}{\rho c_0^2} q^2 - v \left(1 + v_0 s \right) b_i^*, \quad i = 2,3 \end{split}$$

and

$$\begin{split} \Delta &= -b_2^* \left(r_1 s_3 - r_3 s_1 \right) + b_3^* \left(r_1 s_2 - r_2 s_1 \right), \quad \Delta_1 = q_1 \left(b_3^* s_2 - b_2^* s_3 \right) \\ \Delta_2 &= -b_3^* s_1 q_1, \quad \Delta_3 = b_2^* s_1 q_1, \quad q_1 = p - F_1 \left(r_7 s \right) \\ s_1 &= -\left[\frac{\left(\mu + \frac{v T_0 p}{2}}{\rho c_0^2} q p_1^2 + q^3 \frac{\mu - \frac{v T_0 p}{2}}{\rho c_0^2} \right], \quad r_1 = \frac{2\mu q^2 p_1}{\rho c_0^2} \\ s_i &= \frac{2\mu q p_i}{\rho c_0^2}, \quad r_i = p_i^2 - \frac{\lambda}{\rho c_0^2} q^2 - v \left(1 + v_0 s \right) b_i^*, \quad i = 2, 3. \end{split}$$

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Case II: Loading along radial direction

At the surface z = 0, load $F_2(r, t)$ applied in the radial direction are

$$\sigma_{x=} = 0, \quad \sigma_{rz} = -F_2(r,t), \quad T = 0.$$

The expressions given by (2.22) - (2.26) with Δ_i replaced by Δ_i^* where

$$\Delta_1^* = p \left(b_3^* s_2 - b_2^* s_3 \right) + F_2(r, s) \left(r_2 b_3^* - r_3 b_2^* \right)$$

$$\Delta_2^* = b_3^* \left(-r_1 F_2(r, s) - p s_1 \right), \quad \Delta_3^* = b_2^* \left(r_1 F_2(r, s) + p s_1 \right).$$

In order to determine the stress function and displacement because of a concentrated force Dirac delta $\frac{\rho_0\delta(r)}{2\Pi r}$, its hankel transform defined by $F_1(r,s) = F_2(r,s) = \frac{\rho_0}{2\Pi}$ must be used.

For $\alpha = 1.0$ and p = 0, the problem reduces to axisymmetric problem in generalized thermoelasticity.

4. NUMERICAL RESULTS

The calculations will be hold on the surface y = 1.0 at t = 1.0. Figures 1-6 with p = 2.0, show the results for v_z, t_{zz} and T in context of Green Linday theory with following values:

$$\begin{split} F &= 6.9 \times 10^{11} \, [\text{ dynel } \text{cm}^2] \,, \quad \sigma = 0.33, \quad \rho = 2.7 [\text{callgms0}] C \\ \text{C}_p &= 0.236 \, [\text{callGms}^\circ \text{C}] \,, \quad K^* = 0.492 \, [\text{callcms}^\circ \text{C}] \,, \quad v = 0.007 \\ T_0 &= 20 \, [^\circ \text{C}] \,, \eta = 1, \quad \mu = \frac{F}{2\eta(1+\sigma)}, \quad \lambda = \frac{F\sigma}{\eta(1+\sigma)(1-2\sigma)} \end{split}$$

corresponding to isotropic elastic mediumn (Sharma, 2005). By using two relaxation times generalized theory of (Lindsay and Green) by taking $\tau_0 = 0.02$, $\vartheta_0 = 0.03$ we get the solution from the grapical results such as:

(a) $\alpha = 0.2$, weakly conductive, (GTHIS-WC),

(b) $\alpha = 1.0$, normal conductive, (GTHIS-NC),

(c) $\alpha = 1.8$, strongly conductive, (GTHIS-SC).

Generalized thermoelastic medium with HIS (GTHIS):

(d) $\alpha = 0.2$, weakly conductive, (GTWHIS-WC),

(e) $\alpha = 1.0$, normal conductive, (GTWHIS-NC),

(f) $\alpha = 1.8$, strongly conductive, (GTWHIS-SC).

Generalized thermoelastic medium without HIS (GTWHIS):

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5. DISCUSSIONS

Normal direction.

Fig. 1 analysed the variations of (v_z) with (r) distance has appreciable effect in the bandgap $0 \le r \le 2.0$. The effect of HIS and thermal conductivity are clearly visible in the range. The effect of these factors however decline with increase in radial distance r in the band gap of $7.0 \le r \le 10.0$, the effect of HIS and thermal conductivity is almost negligible.

Fig. 2 analysed the variations of (t_{zz}) with (r). Normal force stress increases sharply for weak conductivity from $0 \le r \le 5.8$ with HIS whereas without HIS the value of (t_{zz}) are very less. With HIS variation of (t_{zz}) are more oscillatory for weak and highly conductive material whereas the variation is close to zero for normal conductivity.

Fig. 3 analysed the variations of (T) with (r). This fig shows that without HIS the variation of (T) are oscillatory in nature also these variation are similar for highly conductive, normal conductive and weakly conductive material. It is also observed that the value of (T) for a particular conductive material are very less in the presence of HIS.

Radial direction.

Figure 4 analysed that the value of (v_z) decreases sharply in the band gap of $0 \le r \le 2.0$. The decrease is maximum for normal conductive material. Without HIS value of (v_z) are opposite in nature for highly conductive and normal conductive material.

The variations of $(t_{=x})$ with (r) distance is analysed from figure 5 with HIS the value of (t_{xz}) are mirror image of weakly conductive and highly conductive material in the band gap of $0 \le r \le 3.0$, where as the value of $t_{=2}$) are highly oscillatory for normal conductive material than highly conductive material without HIS.

Figure 6 analysed the variation of (T) with (r). In the presence of HIS the value of T are oscillatory for normal conductive and highly conductive material.

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6. FIGURES



Figure 1: Variation of Normal Displacement vz with distance r (Loading along normal direction)



Figure 2: Variation of Normal force stress tzz with distance r (Loading along normal direction)

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Figure 3: Variation of Temperature Distribution T with distance r (Loading along normal direction)



Figure 4: Variation of Normal force stress tzz with distance r (Loading along radial direction)



Figure 5: Variation of Temperature Distribution T with distance r (Loading along radial direction)

7. CONCLUSIONS

We conclude that Hydrostatic initial stress and thermal conductivity shows critical impact on all the quantities, for both normal and radial source the variation in quantities are almost same with some magnitude difference. In the absence of HIS the variation in (T) are oscillatory in nature.

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